An Investigation of Model Risk in a Market with Jumps and Stochastic Volatility

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EMLYON Workshop in Quantitative Finance and Insurance
Ecully, France
November 27, 2015
The Context

The Question :
→ How much can I lose because I use the wrong model?

Motivations
⊲ Key question for financial institutions (Banks) and bodies regulating them
⊲ The actual measurement of model risk for popular OTC products
⊲ Its eventual reduction
⊲ A realistic market setup
The Context

**The financial universe**

- In their trading books, banks are holding large positions in variance swaps and forward-start options,
- Written on equity indices with liquid vanilla option markets (European call and put options).
- Jumps and stochastic volatility in equity markets

**Chosen markets:**

- Eurostoxx 50 in EUR (SX5E) and S&P 500 in USD (SPX)
- **Data:** Exchange data for call/put option quotes, Bloomberg data for rates and dividends and interbank data for variance swap quotes
Main findings

- Evidence of model risk for variance swaps (100bp irrespective of the maturity)
- Model risk of forward-start options can be reduced by adding variance swaps to the set of calibration instruments
- Insights about the determinants of model risk for the considered instruments
- New pricing formulae obtained along the way
The financial framework

The financial setup

- $S$ the risky asset (stock or equity index), $n$ the money-market account
- trading in continuous time (no arbitrage), $T < +\infty$ the final time horizon
- $d$ the dividend yield paid by $S$, $r$ the risk-free rate earned by $n$
- $Q$ a risk-neutral measure associated with $n$ as numéraire
- $Y$ is the log-return of $S$, that is $Y_t = \ln \left( \frac{S_t}{S_0} \right)$ and $S_t = S_0 e^{Y_t}$

Characteristic function of $Y$

$$\Psi(u, t) = \mathbb{E}^Q \left[ e^{iuY_t} \right]$$  \hspace{1cm} (1)

Martingale constraint on the dynamics of $Y$ under $Q$

$$\mathbb{E}^Q \left[ e^{Y_t} \right] = e^{(r-d)t}$$  \hspace{1cm} (2)
The financial framework

**Instruments (with a liquid market)**

The **forward** price of $S$ for maturity $T$ is obtained as

$$F_{0}^{T} = \mathbb{E}^{Q} [S_{T}] = S_{0}e^{(r-d)T}$$  \hspace{1cm} (3)

The value at $t=0$ of a **call option** with maturity $T$ and strike $K$ is

$$C_{0}(K, T) = B(0, T)\mathbb{E}^{Q} [[(S_{T} - K)^{+}]]$$  \hspace{1cm} (4)

The value at $t=0$ of a **put option** with maturity $T$ and strike $K$ is

$$P_{0}(K, T) = B(0, T)\mathbb{E}^{Q} [[(K - S_{T})^{+}]]$$  \hspace{1cm} (5)
The financial framework

Instruments (that may lead to model risk)

The value at \( t = 0 \) of a **variance swap** with maturity \( T \) (long realized variance) is

\[
VS_0 = N_{var} B(0, T) \left( \frac{1}{T} \mathbb{E}^Q \left[ [Y]_T \right] - K_{var} \right)
\]  

(6)

The value at \( t = 0 \) of a **forward-start call** with maturity \( T \) and tenor \( T - T_0 \) is

\[
CF_0(k, T_0, T) = N_{FS} \times B(0, T) \mathbb{E}^Q \left[ \left( \frac{S_T}{S_{T_0}} - k \right)^+ \right]
\]  

(7)

\( N_{var} \) and \( N_{FS} \) are the notional amounts of the contracts.
Model risk

Let $Z$ be a financial contract written on $S$

- $Z_T$ its final value or payoff
- $Z_T$ is a function of $S_T$ (European derivative) or of the path \{\(S_t, t \in [0, T]\)\}

In the approach of \textsc{Cont} (2006), the model risk of $Z$ is

$$
\varrho(Z_0) = \sup_{M \in \mathcal{M}} \left\{ B(0, T)\mathbb{E}^{Q_M}[Z_T] \right\} - \inf_{M \in \mathcal{M}} \left\{ B(0, T)\mathbb{E}^{Q_M}[Z_T] \right\}
$$

- $\mathcal{M}$ a set of models calibrated to a set of liquid instruments.
- $Q_M$ is the risk-neutral measure when model $M$ has been chosen.
- In the paper the set of models $\mathcal{M}$ gathers 21 models separated in two groups.
Model risk

\[ \varrho(Z_0) = \sup_{M \in \mathcal{M}} \left\{ B(0, T)E^{Q_M}[Z_T] \right\} - \inf_{M \in \mathcal{M}} \left\{ B(0, T)E^{Q_M}[Z_T] \right\} \]

- the measure \( \varrho(Z_0) \) answers the initial question
- clear interpretation in monetary units
- can lead to capital requirements
- no need to choose an \( a \ priori \) model
The set of models : Group 1

**Group 1** : the dynamics of $Y$ is defined by means of stochastic differential equations (SDE).

- (H) is the standard Heston model
- (DH) is a two factor generalization of (H)
- (HJ) is the combination of the dynamics of (H) with independent price jumps

<table>
<thead>
<tr>
<th>No.</th>
<th>Acronym</th>
<th>Name</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>H</td>
<td>Heston</td>
<td>Heston (1993)</td>
</tr>
<tr>
<td>2</td>
<td>DH</td>
<td>Double Heston</td>
<td>Christoffersen et al. (2009)</td>
</tr>
<tr>
<td>3</td>
<td>HJ</td>
<td>Heston with jumps</td>
<td>Bates (1996) and Bakshi et al. (1997)</td>
</tr>
</tbody>
</table>

Table 1. Models in Group 1.
The set of models: Group 2

**Group 2**: the dynamics of $Y$ is built as a time-changed Lévy process (TCLP) that is a Lévy process $X$ (char. exponent $\phi_X$) combined with a change of time $\tau$ (char. exponent $\phi_{\tau}$).

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Name</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>VG</td>
<td>Variance Gamma</td>
<td>Madan and Seneta (1990)</td>
</tr>
<tr>
<td>CGMY</td>
<td>CGMY</td>
<td>Carr et al. (2002)</td>
</tr>
<tr>
<td>Meix</td>
<td>Meixner</td>
<td>Schoutens and Teugels (1998)</td>
</tr>
<tr>
<td>GH</td>
<td>Generalized Hyperbolic</td>
<td>Banrdorff-Nielsen (1977)</td>
</tr>
<tr>
<td>NIG</td>
<td>Normal Inverse Gaussian</td>
<td>Banrdorff-Nielsen (1997, 98)</td>
</tr>
</tbody>
</table>

Table 2. Driving Lévy processes used in the paper.
The set of models: Group 2

A stochastic time change $\tau$ is defined as the integral of a positive stochastic process $y$ known as the rate of time change.

$$\tau(t) = \int_0^t y(s)ds$$  \hspace{1cm} (9)

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<thead>
<tr>
<th>Acronym</th>
<th>Name</th>
<th>References</th>
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<td>CIR</td>
<td>Integrated CIR</td>
<td>Carr et al. (2003) and Cox et al. (1985)</td>
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<tr>
<td>OUG</td>
<td>Integrated OU-Gamma</td>
<td>Barndorff-Nielsen and Shephard (2001)</td>
</tr>
<tr>
<td>OUG</td>
<td>Integrated OU-Inverse Gaussian</td>
<td>Barndorff-Nielsen and Shephard (2001)</td>
</tr>
</tbody>
</table>

Table 3. Stochastic clocks used in the paper.
The set of models : Group 2

The dynamics and characteristic function of $Y$ for models in Group 2

\[ Y_t = \ln \left( \frac{S_t}{S_0} \right) = (r - d)t + X_{\tau(t)} - \omega_{\tau}(t) \]  

\[ \Psi(u, t) = \exp \left[ iu ((r - d)t - \omega_{\tau}(t)) + \phi_{\tau} (-i\phi X(u, 1), t, 1) \right] \]

\[ \omega_{\tau}(t) = \phi_{\tau} (-i\phi X(-i, 1), t, 1) \]
Variance swaps and the log-contract

At maturity, the payoff of a variance swap is $N_{var} \times \left( \frac{1}{T} [Y]_T - K_{var} \right)$ so that the fair variance strike is

$$K_{var}^{fair} = \frac{1}{T} \mathbb{E}^Q [[Y]_T]$$

(11)

When there is no jumps

$$\mathbb{E}^Q [[Y]_T] = 2 \left( (r - d)T + \mathbb{E}^Q [-Y_T] \right)$$

(12)

For models based on time-changed Lévy processes, Carr et al. (2012)

$$\mathbb{E}^Q [[Y]_T] = Q_x \left( (r - d)T + \mathbb{E}^Q [-Y_T] \right)$$

(13)

Following, Neuberger (1994) and Demeterfi et al. (1999), the log-contract is obtained as

$$\mathbb{E}^Q [-Y_T] = -(r - d)T + \frac{1}{B(0, T)} \left[ \int_0^{F_0} \frac{P_0(K, T)}{K^2} dK + \int_{F_0}^{\infty} \frac{C_0(K, T)}{K^2} dK \right]$$
Forward-start options

The payoff of a forward-start call with maturity $T$ and tenor $T - T_0$ is

$$CF_T(k, T_0, T) = N_{FS} \times \left( \frac{S_T}{S_{T_0}} - k \right)^+$$

Its time $t = 0$ value is

$$CF_0(k, T_0, T) = N_{FS} \times e^{-rT}E^Q \left[ \left( \frac{S_T}{S_{T_0}} - k \right)^+ \right]$$

To compute the risk-neutral expectation using a FFT method, we need the FCE that is the CE of the forward log-return $\ln \frac{S_T}{S_{T_0}} = Y_T - Y_{T_0}$ seen from current time $t = 0$

$$\phi_{T_0, T}(u) = \ln E^Q \left[ e^{iu \ln \frac{S_T}{S_{T_0}}} \right] = \ln E^Q \left[ e^{iu(Y_T - Y_{T_0})} \right]$$

$\phi_{T_0, T}$ can be obtained in closed-form for models under scrutiny.
The set of models

<table>
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<tr>
<th>No.</th>
<th>Acronym</th>
<th>Group</th>
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</table>

Table 4. List of all models tested in the study.

- new formula for variance swaps: ⭐
- new formula for forward-start options: ♦
Model risk measures for variance swaps

Figure 1: *Left panel*: measures of model risk for variance swaps on SPX and SX5E. *Right panel*: measures of model risk for variance swaps combined with their hedge in otm options. The variance notionals are $N_{var} = 10000$ USD or EUR. *Red* and *blue* lines are respectively for SPX-2008 and SPX-2012. *Green* and *black* lines are respectively for SX5E-2008 and SX5E-2012.
The reduction of the model risk of Forward-start options

Figure 2: Measures of model risk for forward-start call options. Models are calibrated to vanilla options only (Left panel) or jointly calibrated to vanilla options and variance swap quotes (Right panel). Red, blue, green and black lines are respectively for SPX-2008, SPX-2012, SX5E-2008 and SX5E-2012.

The considered forward-start options have tenor $T - T_0 = 1$ year, relative strike $k = 1.25$ and notional $N_{FS} = 10000$ USD or EUR.
The determinants of model risk

A new question: which feature of our models creates model risk?

To answer it, we introduce the Average Proportion of Model Risk (APMR). It reflects the part of the model risk that is explain by a given feature.

\[
APMR^j = \frac{1}{N} \sum_{i=1}^{N} \frac{\rho_i^j}{\rho_i^{total}},
\]  

for subgroup $j$ and with $N$ instruments considered
The determinants of model risk

<table>
<thead>
<tr>
<th>Nb models</th>
<th>All Instruments (N = 232)</th>
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<tr>
<td>OUG</td>
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<tr>
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</tr>
<tr>
<td>VG</td>
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<td>CGMY</td>
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<td>KOU</td>
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<td>NIG</td>
<td>3</td>
</tr>
<tr>
<td>GH</td>
<td>3</td>
</tr>
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</table>

Table 5. Average Proportion of Model Risk (APMR) captured by subgroups.
Concluding remarks

Contributions

- Existence of model risk for variance swaps
- They use as additional calibration instruments can reduce the model risk of forward-start options
- The determinants of model risk
- Some formulas

Applications

- Risk management of variance swaps as genuine exotic derivatives
- Choice of calibration instruments
- Internal model risk measurement for capital allocation
Conclusion

- I thank you for your attention
- Questions?
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