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NUMÉRO
2009/01

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March 2009

Cahiers de Recherche
Working Papers

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March 7, 2009

Abstract

This paper addresses comparative statics in the consumer problem under uncertainty. Unlike the majority of research in this area we do not assume univariate preferences. We allow a second good to enter preferences independently of the risky good with possibly imperfect substitutability/complementarity. Also in contrast to standard analysis, we allow in the consumption set the presence of different lotteries from which the consumer may choose subject to budgetary constraints. The methodology we follow is based on the lattice programming approach to the deterministic consumer problem developed by Antoniadou (1996) and Mirman and Ruble (2008), which we extend to take account of the presence of uncertainty. We discuss choice and comparative statics as income increases when desired monotonicity is with respect to expenditure on the risky good, as well as when monotonicity is with respect to First or Second Order Stochastic dominance. We derive sufficient conditions in terms of superextremal variant properties of the expected utility function in the constructed lattices for such monotone comparative statics. Our model encompasses the classical portfolio problem under uncertainty which in our context corresponds to perfect substitutability between the two goods. However, our analysis not only extends the existing univariate analysis to the multivariate setting but it can do so in the absence of differentiability of the utility function, or divisibility of the risky good.

Keywords: Lattice Programming, Choice under Uncertainty, Comparative Statics

JEL Classification Codes C61, D11, D81

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†We would like to thank the Department of Economics, Australian National University, and EM Lyon for their hospitality during this research project. We would also like to thank participants at a seminar at the ANU for their comments and suggestions.

1 Introduction

The techniques of comparative statics are important tools used in many scientific endeavors. Comparative statics yields insights into the effect of exogenous changes in the environment on optimal choices. This is especially important in economics research since much empirical evidence is generated and can be explained by models of comparative statics. Indeed, monotone comparative statics is at the heart of understanding much of the empirical evidence. The prototype model of this analysis in economics is demand theory where the demand function is derived from the maximization of utility subject to a budget constraint. Moreover income effects or normality play an important role in analyzing the empirical data. In order to study this comparative statics problem it is usually necessary to assume that the there is enough differentiability as well as a unique optimizer. These are used in order to employ the implicit function theorem as well as other topological methods to derive the monotone comparative statics. This approach yields a local and therefore a narrow version of monotone comparative statics. The assumptions underlying the analysis are very restrictive; for example the assumptions do away with the possibility of analyzing models of discrete choice, non-differentiable choices, and choices that form a correspondence. Ideally, it would be useful to derive conditions for monotone comparative statics in situations that are more general and which are a closer fit to the problems being studied; in particular, to have global rather than local monotone comparative statics.

The use of lattice methods has been introduced into the comparative statics literature to derive general conditions for monotone comparative statics. For example, to be able to employ objective functions that are more general than differentiable functions as well as constraint sets that are more general then convex sets. The work of Veinott [13], and LiCalzi and Veinott [8] analyzes a very general set of superextremal functions that yield monotone comparative statics under suitable monotonicity of the constraint sets. In particular, they employ various types of superextremal functions which, when combined with a strong kind of monotonicity between sets, yield variants of monotonic comparative statics. These methods are not related to the topological methods of the differentiable comparative statics but are entirely order based, so that the notion of differentiability does not enter. Only the notion of "bigger than" applies. This idea is suggested by the lattices induced by Euclidean spaces, where it is necessary to define the notions of "sup" and "inf" of two points. Although suggested by the Euclidean lattice, the method is not limited to these lattices. In fact, the method encompasses a wide variety of spaces and lattices and is applicable to a large assortment of monotone comparative static problems. Thus, the Veinott method supports a more general approach to the monotone comparative statics than does the use of Euclidean lattices. Moreover, the Euclidean lattice is not applicable, in general, for these methods to apply in choice problems since budget constraints are not generally compatible with Euclidean lattices. It turns out that lattices consistent with budget constraints must be used in order to employ Veinott's method.

For the economics problem of a consumer, the archetypical economics problem motivated Antoniadou [1] to introduce a "value lattice" to employ Veinott's method for the case of budget constraints. This method is even more widely applicable as it can be extended to nonlinear constraint sets, as indicated in this paper. The value order was introduced to make the budget sets consistent with Veinott's theorem i.e., to allow for monotone comparative statics when utility is maximized subject to a budget constraint. In fact, it was shown by Antoniadou [1], and [2] and Mirman-Ruble [10] that when using Antoniadou's value lattices the monotone comparative statics common in the economics literature can be extended to a wide variety of situations not covered by the traditional methods, by using these nontraditional lattice theory methods. Mirman and Ruble introduced a version of the value lattice that is particularly suited to non-convex preferences. In all these applications the optimal choice may be a set or a correspondence, thus making the comparative statics much richer, and the notion of monotonicity much more varied in that it now must take account of the ordering of sets. Antoniadou [1] introduced the notions of strong, pathwise and weak normality, to correspond to different notions of choice set comparability. In general, in order to study the comparative statics of a general maximization problem account must be taken of how the optimal sets are ordered, and what kind of monotonicity is thus enabled.

The use of lattices and the theorems of Veinott are much more powerful than have been used in the literature. Lattice programming has been applied to the more general constrained maximization problems, i.e., without differentiability, under certainty. In particular, in the certainty case the monotone comparative statics emphasizes monotonicty in one choice variable, which is chosen on a subset of the real line, i.e., a chain with the natural order. Suppose that the choice variable cannot be defined on a chain, but is defined on a lattice so that not all points are comparable. The study of this problem with the usual tools of comparative statics is inconceivable. However, these problems arise naturally in economic constrained maximization problems, for example decision making under uncertainty. This problem of choice under uncertainty is a natural extension of the certainty demand function problem of an expected utility maximizing agent who must choose subject to a budget constraint. Here one or more of the choices of the consumer is a lottery. Even the constraint set could easily be thought of as a lottery. This adds a depth to the optimization problem and the ensuing comparative statics problem that is not present in the certainty case and has only been dealt with in the economics literature in only very special cases.

In particular, this problem of choice under uncertainty has been studied as a portfolio choice problem where the effect of each good on the utility is exactly the same, i.e., they are perfect substitutes. In this case the utility function is one dimensional rather than multidimensional as in the certainty case. In these portfolio problems, it is assumed that all consumptions are in monetary terms and can, therefore, be added together, an extreme case of pure substitutes. The monotone comparative statics are then derived in the very special case of pure substitutes using the notion of risk aversion of the utility function. However, the problem is much deeper and more encompassing when the utility function is allowed to depend on the effects of each of the goods separately, i.e. there is some degree of complimentarity between the goods. Moreover, using the lattice approach to the comparative statics it is possible that the constraint set is random e.g., income is a random variable. Again a solution to these general comparative static problems under uncertainty is not been challenged except in the special case in the economics literature.

This extension of lattice programming to general models of uncertainty is not as simple as its extension in the case of certainty. First, there are several objectives that may be considered each leading to a different lattice. The set of objects of choice is not in general a chain but has a lattice structure. These underlying lattices must be dealt with in order to optimize as well as to study the monotonicity of the comparative statics. For example, monotonicity may be in the form of the lattice representing first order stochastic dominance, second order stochastic dominant, or the monotone likelihood ratio property. Even in the case of expected utility the objective of the consumer is different with each criterion and the comparative statics must take account of these differences. Different notions of comparability and monotonicity give rise to different notions of comparative statics. The underlying lattices that are consistent with Veinott's theorem must be understood. In particular, just a straightforward generalization of the value lattices to uncertainty will not work.

Finally, especially in the economic choice problem, the price system also plays an important role in the comparative statics. In particular, the choice of a lottery rather that a single point as in the certainty case cannot be described using a linear pricing system because it is not the good itself that is being purchased but the lottery of the good and a pricing system consistent with this choice must be defined. The price system must also be consistent with the lattices that are being used in order to get results that are consistent with Veinott's monotonicty theorem.

In this paper, we introduce the lattice programming structure to problems of optimal choice and monotone comparative statics in a model of choice under uncertainty. This is done by extending Antoniadou's certainty value lattice to the case of uncertainty or lotteries. The expenditure value order of the certainty case, applies directly to the uncertainty problem, providing a link between the two classes of orders (in the certainty and uncertainty case). We further carry out comparative statics analysis when monotonicity of lotteries is with respect to the first order stochastic dominance (FOSD) property and we also show how our results can be extended when comparability is with respect to second order stochastic dominance (SOSD). However, before doing this, we investigate these lattices in the analogous Euclidean space, i.e., product spaces. We show that under both FOSD and SOSD these product spaces are indeed lattices. However as in the certainty case they are not in general consistent with strong budget set orderings and therefore are not well suited to deal with Veinott's comparative statics result. It is then necessary to introduce value lattices that are consistent with the FOSD and the SOSD lattices and budget constraints. It remains to show that the conditions needed to apply Veinott's monotone comparative statics theorem is applicable to the expected utility case. In particular, Veinott and Veinott and LiCalzi posit several properties of the objective function. The two types of functions that we mainly employ are variants of superextremal (SE) functions and are useful for different notions of monotoncity. There is strictly (SSE) and lattice (LSE) superextremal. These functions are ordinal in nature. We relate the properties of the expected utility and the state utility. It is not in general the case that the expected utility is LSE if and only if the state function is LSE. However there is a case when that is true and that is in the case of supermodular functions in the case of the product or Euclidean lattice. Supermodular functions are cardinal functions. Thus comparisons of the two types of functions are difficult to make.

After studying the relationship between expected utility and the state utility functions for SE-type functions, we introduce a stochastic value lattice for the general stochastic choice problem when the set of lotteries are ordered by FOSD. We show that they are actually lattices and can be used to invoke Veinott's theorem. However, it is necessary to attach to the maximization problem a nonlinear pricing function in order that the maximization problem of the consumer is well defined. For the FOSD case these price functions are interesting because they satisfy expected value restrictions due to their consistency to FOSD. However even with this property the prices associated with the lattice of lotteries presents a problem. For inconsistent lotteries may yield prices that do not satisfy the budget constraint. However there are several avenues to dealing with this problem and we show how Veinott's theorem can be applied. We also study SOSD, again using a set of prices that are compatible with the SOSD lattice again yielding results using Veinott's theorem.

2 Background

In this section we provide a brief review of existing results on the application of lattice programming to comparative statics of the consumer problem under certainty as they relate to the subject of this paper, and we also provide a brief background on the lattice methods used in the paper. The reader may refer to our earlier work (Antoniadou [1], [2] and Mirman and Ruble [10]) for further material, and also to Li Calzi and Veinott [8], Veinott [13], Milgrom and Shannon [9] and Davey and Priestly [6] for more technical background.

2.1 Lattices and comparative statics

Comparative statics may be approached, as it is mostly done in economics, with the use of topological tools. But this method is not without its, arguably important, limitations. In order to apply the main topological tool, the implicit function theorem, to an optimum solution, it is necessary that the problem be *smooth enough*. This framework therefore does not apply in the absence of differentiability (as in the case of Leontieff preferences), or continuity (as in the case of discrete choices), or in the presence of non-convexities which may invalidate the conditions on which the theorem is applied to. All this on purely technical rather than economic grounds. Perhaps more importantly, even granted the assumptions on which the use of the implicit function theorem is enabled, there are still many problems which do not avail themselves in any straightforward manner to the methodology. The subject matter of this paper, comparative statics of multivariate choice under uncertainty, being a good case in point.

An alternative approach to comparative statics, using tools of lattice programming, does not impose these restrictions. Using order-based properties, it exploits the natural order relationships that are inherent to the optimization problem, something which the topological approach does not do. The methods can be used in the presence of non-continuity, discrete and multi-valued choice. The ensuing monotone comparative statics results are global results. The application of these tools in the context of income effects in the problem of consumer choice has been formalized and illustrated in our earlier papers, Antoniadou [1], [2] and Mirman and Ruble [10].

One way to understand the intuition behind the lattice programming method is by relating it to the revealed preference approach. Consider the example of consumer choice, and suppose that we wish to study when the consumption of one of the goods, say good y, is nondecreasing in income. Let X and X' be any two bundles that lie in a low and high income budget set respectively, with the restriction that X has a greater quantity of good y than X'. We need to ensure that these cannot be optimal under their respective budget sets. We can do so if we can identify two alternative bundles, W and Z, lying in the low and high budget sets, and with the same amounts of good y as X' and X respectively, and such that if X is revealed (weakly) preferred to W, then X' cannot be revealed preferred to Z :

$$u(X) \geq u(W) \Rightarrow u(X') \not> u(Z) \tag{1}$$

where $u(\cdot)$ is the utility function. Namely, if a high-good y bundle is preferred at low income, then this preference must arise at high income as well (and if a low-good y bundle is preferred at high income, then this preference must arise at low income as well). The issue is how to arrive at these alternative bundles (which to be meaningful must be chosen systematically), and how to establish the ordering of preference. The answer to the first question, provided by Antoniadou [1] using the lattice programming methods, is the identification of W, Z with the meet $(X \wedge X')$ and join $(X \vee X')$ of X, X' in an *appropriately constructed lattice*. The answer to the second question is in the superextremal variant properties of functions as employed in lattice programming.

Under certainty, the consumer choice problem is:

$$\max_{x \in \mathbb{R}^n_+} u(x) \text{ s.t. } p \cdot x \leq I \tag{2}$$

As it is habitual to order the consumption set, $\mathbb{R}^n_+$, with the Euclidean order $\leq_\varepsilon$, the temptation is to embed this problem in the lattice $\left(\mathbb{R}^n_+, \leq_\varepsilon\right)$. However, this first approach is not fruitful. The lattice method requires ordering the consumption space in a manner that is consistent with the problem being studied. This insight led Antoniadou [1] to introduce a new kind of partial order adapted to the context of income effects, the direct value order. This order reflects the fact that it is not whether one bundle is larger than another in the Euclidean sense, but rather whether the bundle reflects a higher consumption of one of the goods, and whether this bundle is more valuable at market prices.

Definition 1 *Direct value order (Antoniadou [1]): Let* $X, Y \in \mathbb{R}^n_+$*, then* $X \leq_{d.v.(p)} Y$ *if and only if* $X \leq_l Y$ *and* $p \cdot X \leq p \cdot Y$*.*[1]

This partial order defines an alternative lattice on the consumption set, $\left(\mathbb{R}^n_+, \leq_{d.v.(p)}\right)$, to which standard lattice programming techniques can be usefully applied to establish normality, for instance in the cases of Leontieff-like preferences, of certain non-convex preferences, and so forth.[2] We next review the relevant techniques.

2.2 Lattice programming tools for comparative statics

We summarize the relevant lattice programming tools. Let $(\mathcal{X}, \leq_{\mathcal{X}})$ be a lattice. The following two sets of concepts, one related to sets and the other to functions, are important for the main comparative statics theorem:

Definition 2 *(Set orders - Veinott [13]) Let A and B be two sets in* $\mathcal{X}$*:*

Strong set order: $A \leq_a B$ *iff for all* $X \in A$, $X' \in B$, $X \wedge X' \in A$ *and* $X \vee X' \in B$,

Chain-lower-than: $A \leq_c B$ *iff* $A \leq_a B$ *and all* $X \in A$, $Y \in B$ *are comparable,*

Strongly-lower-than: $A \leq_s B$ *iff for all* $X \in A$, $Y \in B$ $\;X \leq_{\mathcal{X}} Y$.

Definition 3 *(Supermodular -Superextremal properties) Let* $f : \mathcal{X} \to \Re$ *be a real-valued function*

f is supermodular (SM) iff, for all $X, X' \in \mathcal{X}$,

$$f(X \vee X') + f(X \wedge X') \geq f(X) + f(X')$$

f is lattice superextremal (LSE) iff, for all $X, X' \in \mathcal{X}$,

$$f(X) \geq (>)\, f(X \wedge X') \Rightarrow f(X \vee X') \geq (>)\, f(X').$$

f is strictly superextremal (SSE) iff, for all incomparable $X, X' \in \mathcal{X}$,

$$f(X) \geq f(X \wedge X') \Rightarrow f(X \vee X') > f(X').$$

[1] Here, $\leq_l$ is a lexicographical order. Unless otherwise specified this is taken with respect to the natural indexing. With two goods, if the natural indexing is not used, this is indicated as: $(x, y) \leq_{l(y)} (x', y')$ if either $y < y'$ or $y = y'$ and $x \leq x'$.

[2] The straightforward idea that lattice methods can be applied much more broadly by identifying appropriate orders has yet to gain much ground. We are not aware of other examples outside the work already cited, except the work of Vives [14] who uses a simple transformation on payoffs to establish that Cournot duopoly is a supermodular game.

The SM property is cardinal, whereas the LSE and SSE properties are ordinal. A SM function is LSE. An increasing transformation of an SM function is LSE (while an affine transformation of an SM function is SM).

We say that a real-valued function is increasing on a poset $(X, \leq_X)$, $f : X \rightarrow \Re$, if $x \leq_X x' \Rightarrow f(x) \leq f(x')$ and $x <_X x' \Rightarrow f(x) < f(x')$. We say that f is non-decreasing if $x \leq_X x' \Rightarrow f(x) \leq f(x')$.[3] An increasing function on a lattice $(X, \leq_X)$ is SSE (and thus LSE, but not necessarily SM). This may render the LSE property deceptively impotent in consumer problems, since it may be argued that a function being increasing is a fairly innocuous assumption in consumer choice. However, it is a property that does not buy us much in the way of comparative statics since it offers no restrictions useful for comparative statics. Yet, such reasoning is deceptive because it extrapolates and generalizes something which is special to the Euclidean lattice (and to a lesser extent product lattices more generally). Outside the Euclidean lattice, the property itself need not be as innocuous, and the comparative statics it enables need not be trivial. In a non-Euclidean lattice, the up set of a point may contain points that are not in the Euclidean up set, and it may not contain points that are in the Euclidean up set (its intersection with the Euclidean up set may even be empty). Therefore, the assumption that all points in an non-Euclidean up set are (weakly) preferred no longer corresponds to the classical notion of desirability, and it may be quite restrictive indeed.

The reason the ordinal lattice theoretic properties are so important is due to the following main comparative statics theorem, which establishes equivalence between the LSE (SSE) property and the ordering of optimum sets:

Theorem 4 *(Li Calzi and Veinott [8][4]) Let* $f : \mathcal{X} \rightarrow \mathcal{R}$ *and* $A, B \subset \mathcal{X}$*. Then*
f is LSE iff $\arg\max_A f \leq_a \arg\max_B f$ *for all* $A \leq_a B$,
f is SSE iff $\arg\max_A f \leq_c \arg\max_B f$ *for all* $A \leq_a B$,
f is MSE iff $\arg\max_A f \leq_m \arg\max_B f$ *for all* $A \leq_a B$,
whenever $\arg\max_B f$, $\arg\max_A f \neq \emptyset$.

For comparative statics subject to budget constraints, the most frequent application of the theorem relates to sufficiency: when f satisfies a variant of the superextremal condition, the behavior of the set of optimizers is determined. The necessity part requires that we allow restricted constraint sets that may not naturally arise in the problem. The question of determining comparative statics thus becomes one of determining a lattice in which strong constraint set comparability is enabled, and thus the superextremal variant conditions can be used.

[3] The up set of a point is the set of points that are weakly better: $x \uparrow = \{x' \in X \mid f(x') \geq f(x)\}$. The down set is defined analogously.

[4] Milgrom and Shannon [9] also have the first part of the theorem.

2.3 Value orders and the consumer problem under certainty

The use of lattice methods in the comparative statics of consumer choice can now be illustrated. We focus on the two good case,

$$\max_{(x,y)\in\mathbb{R}^2_+} u(x,y) \text{ s.t. } p_x x + p_y y \leq I. \tag{3}$$

A priori, one might reason as follows in order to characterize the income effects with respect to good y. If the budget sets $B(I) = \left\{(x,y) \in \mathbb{R}^2_+ \mid p_x x + p_y y \leq I\right\}$ are increasing in I (with respect to set ordering $\leq_a$), then by Theorem 4, if u is LSE (SSE), the set of optimal choices is increasing (with respect to a set ordering $\leq_a$ ($\leq_c$)), which would correspond to normality of good y.

It is natural to try to implement this logic by considering the lattice $\left(\mathbb{R}^2_+, \leq_\varepsilon\right)$.[5] This yields a superficially surprising result: any nondecreasing utility function u is LSE in this lattice. This suggests that the LSE restriction is not informative with regard to normality. In fact this occurs because this lattice is ill-adapted to the problem of consumer choice: the set order $\leq_a$ induced by the Euclidean order does not rank budget sets, i.e. for $I' > I$, $B(I') \ngeq_a B(I)$:

Example 5 *Let* $X = \left(0, \frac{I}{p_y}\right)$ *and* $X' = \left(\frac{I}{p_x}, 0\right)$*, so* $X \in B(I)$ *and* $X' \in B(I')$*. Then,* $X \wedge X' = (0,0) \in B(I)$*, but* $X \vee X' = \left(\frac{I}{p_x}, \frac{I}{p_y}\right) \notin B(I')$*. Therefore,* $B(I') \ngeq_a B(I)$*.*

The application of Theorem 4 in $\left(\mathbb{R}^2_+, \leq_\varepsilon\right)$ is thus not informative as to the effect of such shifts in budget sets.[6] This limitation pertains to the lattice, $\left(\mathbb{R}^2_+, \leq_\varepsilon\right)$, and not to the theorem. The problem can be circumvented by embedding the choice problem into more adapted lattices, such as the value lattices introduced by Antoniadou [1], that address the problem of budget set comparability while remaining consistent with the desired comparative static (e.g., increase in y).

Returning to the direct value order, note that in $\left(\mathbb{R}^2_+, \leq_{d.v.(p,y)}\right)$ for two points $X = (x,y)$ and $X' = (x', y')$, assuming that $p_x x + p_y y \leq p_x x' + p_y y'$, the meet and join defined by $\leq_{d.v.(p,y)}$ are,

$$X \wedge X' = \left(x + \frac{p_y}{p_x}\left(y - y \wedge y'\right), y \wedge y'\right) \text{ and} \tag{4}$$

$$X \vee X' = \left(x' - \frac{p_y}{p_x}\left(y \vee y' - y'\right), y \vee y'\right). \tag{5}$$

[5] In what follows, when there is no danger of confusion, as with the Euclidean order, we do not refer to the underlying partial order, and use $\leq$ and $<$ with no subscript. Otherwise, we subscript inequality symbols as necessary.

[6] In fact the problem is deeper than just that pertaining to income effects. A subset of the Euclidean lattice is not a Euclidean sublattice unless it is a *box*. Therefore, a constraint set with budgetary trade-offs cannot be a Euclidean sublattice, and by extension relevant constraint sets cannot be ordered by the strong set order.

The direct value order ranks bundles by both their value and the amount of one of the two goods (here, y). By construction, it solves the problem of budget set comparability, as $p \cdot X \vee X' = (p \cdot X) \vee (p \cdot X')$ (also, $p \cdot X \wedge X' = (p \cdot X) \wedge (p \cdot X')$). That is, in the lattice $\left(\mathbb{R}_+^2, \leq_{d.v.(p,y)}\right)$, $B(I') \geq_a B(I)$ for $I' \geq I$. Then, Theorem 4 can be invoked to conclude that, when u is LSE (SSE), then the optimal choice set is nondecreasing with respect to $\leq_a$ ($\leq_c$) in income. By construction of the value order, this entails that the optimal choice of y is nondecreasing in income, i.e. normal.

Since the lattice approach does not restrict consideration to strictly quasi-concave preferences, Theorem 4 is consistent with multiple optima. This means that the notion of a normal good must be specified. As the (sets of) optima are increasing with respect to $\leq_a$, it seems reasonable to say that good y is normal in the sense that every optimal choice at high income is greater than or equal to some optimal choice at low income, and every optimal choice at low income is smaller than or equal to some optimal choice at high income.[7] And if u is SSE in this lattice, then the sets of optima are increasing with respect to $\leq_c$, which when at least one good is desirable implies the strongly-lower-than comparability, $\leq_s$, which in turn means that good y is strongly normal in the sense that every optimal choice at high income is weakly greater than every optimal choice at lower income.

As a first illustration of the application of lattice methods, consider the commonly used Cobb-Douglas preferences.

Example 6 *Suppose that $u(x, y) = x^a y^b$. Then, u is SSE in $\left(\mathbb{R}_+^2, \leq_{d.v.(p,y)}\right)$ and $\left(\mathbb{R}_+^2, \leq_{d.v.(p,x)}\right)$, so by Theorem 4, both goods are strongly normal.*

Although it is encouraging that lattice methods apply to a standard case, in itself, example 6 is not so useful. The normality is easily established for these preferences with the standard approach to comparative statics or by direct solution. However, suppose that good y is a *discrete* good. In this case, the optimal choice is more tedious to calculate,[8] and the implicit function theorem cannot be used on the first order conditions since it is not possible to differentiate. The lattice approach, on the other hand, is immediate. As $\left(\mathbb{R}_+ \times \mathbb{N}, \leq_{d.v.(p,y)}\right)$ is a sublattice of $\left(\mathbb{R}_+^2, \leq_{d.v.(p,y)}\right)$, $B(I) \leq_a B(I')$ for $I \leq I'$ in $\left(\mathbb{R}_+ \times \mathbb{N}, \leq_{d.v.(p,y)}\right)$ as well. As u is SSE, by Theorem 4, the optimal (discrete) choice of y is nondecreasing in income.[9]

Finally, we note that the direct value order is not the only value order, but one of a broader family of "value" orders which may be more or less suited to particular types of problems. Mirman and Ruble [10] define the radial value order on $\mathbb{R}_+^2$, as follows:

[7] In this case Antoniadou [1] defines a good as *pathwise* normal, corresponding to *pathwise* comparability of sets.

[8] For instance, for example 6 with $a = b$ and $p_x = p_y = 1$, $X^* = \left(I - \left\lceil \frac{I+1}{2} \right\rceil, \left\lceil \frac{I+1}{2} \right\rceil\right)$ when $I \neq 2n+1$, $n \in \mathbb{N}$, and $X^* \in \left\{\left(I - \left\lceil \frac{I+1}{2} \right\rceil, \left\lceil \frac{I+1}{2} \right\rceil\right), \left(I - \left\lceil \frac{I+1}{2} \right\rceil + 1, \left\lceil \frac{I+1}{2} \right\rceil - 1\right)\right\}$ otherwise.

[9] However, in this case we cannot establish that both goods are simultaneously normal.

Definition 7 *Let* $X, Y \in \mathbb{R}^2_+$*, then Radial value order:* $X \leq_{r.v.(p,y)} X'$ *if and only if* $yx' \leq xy'$ *and* $p \cdot X \leq p \cdot X'$*.*

This partial order compares bundles across budget sets and rays from the origin. As such, it is particularly well-adapted to homothetic preferences. As with the direct value order, $\left(\mathbb{R}^2_+, \leq_{r.v.(p,y)}\right)$ is a lattice that satisfies $B(I) \leq_a B(I')$ for $I \leq I'$. Consider the following example of preferences, drawn from Mirman and Ruble [10]:

Example 8 *Suppose that* $u(x, y) = \left(\frac{x}{a_1} \wedge \frac{y}{b_1}\right) \vee \left(\frac{x}{a_2} \wedge \frac{y}{b_2}\right)$*, with* $a_1 + b_1 = a_2 + b_2$*. Then, u is not LSE in* $\left(\mathbb{R}^2_+, \leq_{d.v.(p,y)}\right)$*. However, u is SSE in* $\left(\mathbb{R}^2_+, \leq_{r.v.(p,y)}\right)$*, so by Theorem 4, y is strongly normal.*

These preferences, which are depicted in Figure 1, along with a pair of incomparable points, illustrate the power of the lattice approach: the utility function is neither differentiable, nor quasiconcave. Yet clearly, the goods are normal – a fact which the standard approach does not capture but which can be encompassed by the lattice approach.

While Antoniadou [1], [2], and Mirman and Ruble [10] focus on the consumer problem under linear pricing, it is clear that the method can be extended to address non-linear pricing. This is a useful point for the results of this paper since, in the case of lotteries, linear pricing, is not necessarily a useful assumption. In particular, suppose that the expenditure on y is given by a strictly increasing function, $p(y) : \mathbb{R}_+ \to \mathbb{R}_+$, the inner product function under linear pricing being a special case. Then, the following is a partial order on $\mathbb{R}^2_+$:

Definition 9 *(Expenditure value order): Let* $X = (x, y), X' = (x', y') \in \mathbb{R}^2_+$*. They are comparable with respect to the expenditure value order* $\leq_{ev(p(y))}$*,* $X \leq_{ev(p(y))} X'$*, iff* $p_x x + p(y) \leq p_x x' + p(y')$ *and* $p(y) \leq p(y')$*.*

It is straightforward to establish that $\leq_{ev(p(y))}$ is reflexive and transitive on $\mathbb{R}^2_+$. Antisymmetry follows from the fact that the price function $p(y)$ is strictly increasing and thus uniquely identifies the consumption level of y. Therefore, given $X \leq_{ev(p(y))} X'$, $p(y) \leq p(y')$ implies $y \leq y'$ thus allowing us to define good y, as normal in the same way as under linear pricing.

For two incomparable points X and X' with $p_x x + p(y) < p_x x' + p(y')$ and $p(y) > p(y')$, the meet and join are given by:

$$X \vee X' = \left(x' - \frac{p(y) - p(y')}{p_x}, y\right) \text{ and } \tag{6}$$

$$X \wedge X' = \left(x + \frac{p(y) - p(y')}{p_x}, y'\right) \tag{7}$$

As $x' - \frac{p(y)-p(y')}{p_x} > x \geq 0$, $X \vee X', X \wedge X' \in \mathbb{R}^2_+$ so $\left(\mathbb{R}^2_+, \leq_{ev(p(y))}\right)$ is indeed a lattice. In the particular case where $p(y) = p_y y$, $\left(\mathbb{R}^2_+, \leq_{ev(p(y))}\right) =$

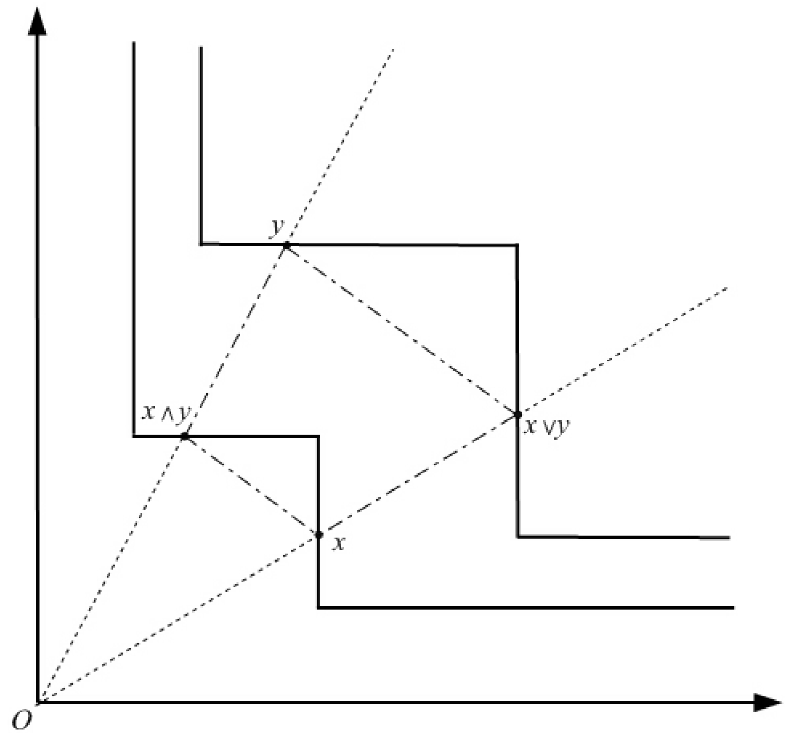

Figure 1: Figure 1

$\left(\mathbb{R}_+^2, \leq_{d.v.(p,y)}\right)$. Moreover, $p_x (X \vee X')_x + p\left((X \vee X')_y\right) = (p_x x + p(y)) \vee (p_x x' + p(y'))$ and $p_x (X \wedge X')_x + p\left((X \wedge X')_y\right) = (p_x x + p(y)) \wedge (p_x x' + p(y'))$ so budget sets are strongly ranked in this lattice and thus Theorem 4 can be applied to give sufficient conditions for good y to be normal under this more general pricing rule.

With nonlinear prices, the configuration of allowable budget sets is greatly broadened. The assumption that the expenditure function $p(y)$ is strictly increasing implies that the budget frontier is downward sloping (in the consumption set), and thus budgetary trade-offs continue to apply. Other than this, it is possible to use, for example, non-convex budget sets (in the case of quantity discounts), which is yet another instance of the lattice method's generality compared to the traditional assumptions.

An example of non-linear prices is illustrated in Figure 2: there is a quantity discount, that is $p(y) = \underline{p}_y$ for $y \leq y^*$ and $\overline{p}_y$ otherwise, with $\overline{p}_y < \underline{p}_y$. The constraint set is neither convex nor differentiable, and the preferences depicted are not differentiable. However, the optimal choice of y increases with income, and can have multiple values ($A1$ and $A2$), and yet this kind of behavior can be captured with lattice methods discussed here.

In the remainder of this paper, we use the intuition from the certainty case to study comparative statics in the consumer problem under uncertainty.

3 Preferences over risk with two goods

Having reviewed the certainty case, we next posit an expected utility maximizer who has preferences over two goods. The first good is deterministic while the second is subject to risk:

$$U(x, \widetilde{y}) = E_{\widetilde{y}} u(x, y) = \int u(x, s) dF_y(s)$$

The domain of U is the consumption set $\mathbb{R}_+ \times \mathcal{F}_y$, where $\mathcal{F}_y$ is the set of distribution functions on $\mathbb{R}_+$. We denote a typical element of the consumption set as either (x, F_y) or as $(x, \widetilde{y})$, where $\widetilde{y}$ is the random variable with distribution function $F_y \in \mathcal{F}_y$. We use the two different notations interchangeably as convenient. The state utility function, u, is defined over $\mathbb{R}_+^2$ and represents the individual's preferences over the realizations of consumption. Our formulation encompasses the perfect substitutes specification $u(x + y)$ as a special case.

Allowing all distribution functions over $\mathbb{R}_+$ in the consumption set is a very general setting. In the existing literature there is typically one distribution function, and much less often two with special independence/dependence assumptions attached to them (and one typically assumed to be background riskiness, and thus not subject to choice). We study such restricted domains elsewhere. In this paper we keep the setting as general as possible. Nonetheless, we need to impose some restrictions on the domain in order to apply Theorem 4. Notice that in the general setting of this paper the quantity of a lottery purchased

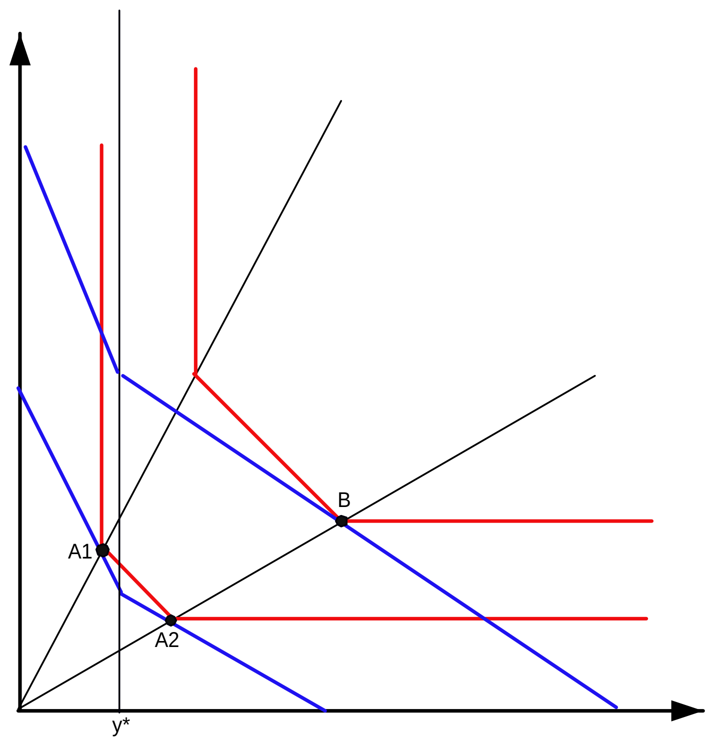

Figure 2: Figure 2

need not be specified, since this possibility is embedded in the availability of a different distribution function.

We study the comparative statics of choice problems with uncertainty in the following section. But doing this requires a notion of what it means for a bundle $(x, \widetilde{y})$ to increase, and in particular what it means for $\widetilde{y}$ to increase. This is simple in the certainty case, and it is relatively straightforward when there is only one underlying distribution function in the consumption set, i.e. all that can change is the *quantity* of the specified distribution function. However, when there are two or more underlying distribution functions, or in the general case, it is important to specify what it means for $\widetilde{y}$ to *increase*, or how different distribution functions are to be compared. For this we use concepts such as First and Second Order Stochastic Dominance (FOSD and SOSD, respectively).

In this section we use FOSD and SOSD in order to form product orders and lattices on the consumption set, analogous to the Euclidean order/lattice in the certainty case. This allows us to formalize results from the univariate or perfect susbtitutes case, to our more general context. We relate FOSD and SOSD to increasing and increasing and concave preferences, respectively and in turn link the lattice theoretic properties of the expected utility function to properties of the state utility function. However, for reasons analogous to those in the certainty case, these product orders cannot accommodate comparative statics in the budget constrained problem. Therefore, in the remainder of the paper we use the intuition from the certainty case in this section to construct value orders appropriate for this richer consumption set, $\mathbb{R}_+ \times \mathcal{F}_y$, that account both for multiple goods and for riskiness.

3.1 The product order and lattice with FOSD

To begin with, we focus on First Order Stochastic Dominance. FOSD is attractive in that it is noncontroversial, having the property that a FOSD change is preferred by all increasing utility functions, in the perfect substitutes case and also in the more general case here, as we establish below. Thus, it offers a fairly noncontroversial definition of an *improvement* or *increase* in the risky good. In the case of a restricted domain, or a consumption set with only one underlying distribution function, changes in the quantity of the risky good, correspond to FOSD changes. Therefore, FOSD offers a particularly compelling notion of an increase in $\widetilde{y}$. It does however come at the cost of relatively restrictive comparability in the general setting. Therefore, we also use different, less stringent, ways of comparing risks.

Let $\leq_{FOSD}$ be defined over $\mathcal{F}_y$ by $F_y \leq_{FOSD} F_{y'}$ if and only if $F_y(s) \geq F_{y'}(s)$ for all $s \in \mathbb{R}_+$, for any $F_y, F_{y'} \in \mathcal{F}_y$.

Lemma 10 *$(\mathcal{F}_y, \leq_{FOSD})$ is a lattice.*

Proof. First, $(\mathcal{F}_y, \leq_{FOSD})$ is a poset. $\leq_{FOSD}$ is a partial order, since $F_y \leq_{FOSD} F_y$. Moreover $F_y \leq_{FOSD} F_{y'}$ and $F_{y'} \leq_{FOSD} F_y$ imply $F_y(s) = F_{y'}(s)$ for all s, and $F_y \leq_{FOSD} F_{y'}$ and $F_{y'} \leq_{FOSD} F_{y''}$ imply $F_y \leq_{FOSD} F_{y''}$. Next, if $F_y, F'_y \in \mathcal{F}_y$ are two incomparable distributions, the join and meet are given by

$[F_y \vee F_{y'}](s) = \min\{F_y(s), F_{y'}(s)\}$, and
$[F_y \wedge F_{y'}](s) = \max\{F_y(s), F_{y'}(s)\}$
(i.e. the component-wise min and max) respectively. $F_y \vee F_{y'}$ and $F_y \wedge F_{y'}$ are both distributions in $\mathcal{F}_y$, so $(\mathcal{F}_y, \leq_{FOSD})$ is a lattice. ■

Note that the $(\mathcal{F}_y, \leq_{FOSD})$ lattice, as defined, has a lowest element which is the degenerate distribution at 0, $F_0 \equiv 1_{\mathbb{R}_+}$. This is not the case if the consumption set were defined to allow for negative consumption realizations of the risky good y.

A first approach to ordering the consumption set is to proceed by analogy with the Euclidean order and use the product order, where x is ordered by the usual order. The *product FOSD* order is defined as $(x, F_y) \leq_{(\varepsilon \times FOSD)} (x', F'_y)$ if and only if $x \leq x'$ and $F_y \leq_{FOSD} F_{y'}$. The consumption set is a lattice with the product FOSD order:

Lemma 11 *$(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{(\varepsilon \times FOSD)})$ is a lattice.*

Proof. Let (x, F_y) and $(x', F_{y'})$ be two points in $\mathbb{R}_+ \times \mathcal{F}_y$. The join and meet are defined as $(x \vee x', F_y \vee_{FOSD} F_{y'})$ and $(x \wedge x', F_y \wedge_{FOSD} F_{y'})$ respectively, and both are elements of $\mathbb{R}_+ \times \mathcal{F}_y$. ■

We call this the *product FOSD lattice*. The analogy with the Euclidean case is not perfect, because $\mathcal{F}_y$ is itself a non-trivial lattice, and not simply a chain. As a result, two bundles may be incomparable even though they have the same quantity of the sure good x. This occurs when the two associated distributions are not ranked by $\leq_{FOSD}$, in which case $F_y \vee F_{y'} \neq F_y, F_{y'}$ and $F_y \wedge F_{y'} \neq F_y, F_{y'}$. If, however, we consider a lattice over some restricted consumption set $\mathbb{R}_+ \times \mathcal{F}_y^C$ where $\mathcal{F}_y^C \subseteq \mathcal{F}_y$ is a chain of distribution functions with respect to $\leq_{FOSD}$, then the analogy is better.

The next question relates to the properties of the expected utility function on the consumption set, i.e. the lattice, $(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{(\varepsilon \times FOSD)})$. We describe how lattice theoretic properties (SM, LSE, SSE), defined on the lattice $(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{(\varepsilon \times FOSD)})$, are linked to the preferences expressed by the state utility u and to the ordering on risks $\leq_{FOSD}$. Since these properties apply over *all* possible lotteries (over the full set $\mathcal{F}_y$), whether U is LSE (SSE etc) or not depends on properties of the expectation operator and the underlying state utility function.

A first simple result is that lattice theoretic properties in the Euclidean lattice are necessary for the corresponding property in the product FOSD lattice:

Lemma 12 *$U(x, \widetilde{y})$ has a lattice property (SM, LSE, SSE) in $(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{(\varepsilon \times FOSD)})$ only if $u(x, y)$ has the corresponding property in $(\mathbb{R}_+^2, \leq_\varepsilon)$.*

Proof. Any pair of (x, y) and (x', y') in $(\mathbb{R}_+^2, \leq_\varepsilon)$, has a corresponding pair $(x, \widetilde{y})$ and $(x', \widetilde{y}')$ in $(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{(\varepsilon \times FOSD)})$ where $\widetilde{y}, \widetilde{y}'$ are two degenerate lotteries with all probability mass at y and y' respectively. Therefore, the corresponding property in $(\mathbb{R}_+^2, \leq_\varepsilon)$ is necessary for any lattice property of the expected utility function in $(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{(\varepsilon \times FOSD)})$. ■

In the case of the SM property, its validity in the Euclidean lattice is not only necessary but sufficient for it to hold in $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{(\varepsilon \times FOSD)}\right)$. This is proven in Lemma 13. The result appears as a subcase in Athey [4], although the focus in that paper is to relate properties of parameters of a distribution function to properties of the expected utility. The direct proof given below highlights the link between lattice properties of the two (choice) spaces, $\left(\mathbb{R}_+^2, \leq_\varepsilon\right)$ and $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{(\varepsilon \times FOSD)}\right)$. This result is special to the product space and does not carry over to the case of the value lattices studied in the following section.

Lemma 13 *U is SM in $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{(\varepsilon \times FOSD)}\right)$ if and only if u is SM in $\left(\mathbb{R}_+^2, \leq_\varepsilon\right)$.*

Proof. $\Rightarrow$ Follows from Lemma 12

$\Leftarrow$ Suppose that u is SM, and let $X = (x, \widetilde{y})$ and $X' = (x', \widetilde{y}')$ be two incomparable points in $\mathbb{R}_+ \times \mathcal{F}_y$ with $x' \geq x$ and $\widetilde{y}' \ngeq_{FOSD} \widetilde{y}$. U is SM if $\int u(x,s)dF_y(s) + \int u(x',s)dF_{y'}(s) \leq \int u(x,s)dF^\wedge(s) + \int u(x',s)dF^\vee(s)$, which can be rewritten as:

$$\int u(x,s)\left[dF_y(s) + dF_{y'}(s)\right] + \int \left[u(x',s) - u(x,s)\right]dF_{y'}(s) \qquad (8)$$
$$\leq \int u(x,s)\left[dF^\wedge(s) + dF^\vee(s)\right] + \int \left[u(x',s) - u(x,s)\right]dF^\vee(s).$$

With the $\leq_{FOSD}$ order, $dF_y(s) + dF_{y'}(s) = dF^\wedge(s) + dF^\vee(s)$ for all s, so the first terms on each side of the inequality are identical. Since u is supermodular, $u(x', s) - u(x, s)$ is either 0 (if $x' = x$), or nondecreasing in s. Therefore, since $F^\vee >_{FOSD} F_{y'}$,

$\int \left[u(x',s) - u(x,s)\right]dF^\vee(s) \geq \int \left[u(x',s) - u(x,s)\right]dF_{y'}(s)$

so (8) holds. ■

This result is special to the SM property and does not extend to the ordinal lattice theoretic properties. We show this by example in the case of the LSE property (the rest follow). Of course since in any lattice the SM property is sufficient for the LSE property Lemma 13 gives a sufficient condition for the expected utility function to be LSE in $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{(\varepsilon \times FOSD)}\right)$.

Example 14 *Consider the points (sublattice) in $\mathbb{R}_+^2 : \{0,1\} \times \{0,1,2,3\}$. Suppose u takes the values $3,2,0,2$ at the point $(0,0),\ldots,(0,3)$ respectively, and the values $2,0,2,3$ at the points $(1,0),\ldots,(1,3)$, respectively. Then u is LSE at these points (there are six incomparable pairs to verify) but not increasing. Let $\widetilde{y} = \{0.5, 0.5; 1, 3\}$ and $\widetilde{y}' = \{0.5, 0.5; 0, 2\}$, so that $F_{y'} <_{FOSD} F_y$. Then $(0,\widetilde{y}), (1,\widetilde{y}') \in \left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{(\varepsilon \times FOSD)}\right)$ are incomparable with $(0,\widetilde{y}) \vee (1,\widetilde{y}') = (1,\widetilde{y})$ and $(0,\widetilde{y}) \wedge (1,\widetilde{y}') = (0,\widetilde{y}')$. Then $U(0,\widetilde{y}) = 0.5 \times 2 + 0.5 \times 2 = 2 > 0.5 \times 3 + 0.5 \times 0 = U((0,\widetilde{y}) \wedge (1,\widetilde{y}'))$, and $U(1,\widetilde{y}') = 0.5 \times 2 + 0.5 \times 2 = 2 > 0.5 \times 3 + 0.5 \times 0 = U((0,\widetilde{y}) \vee (1,\widetilde{y}'))$. Thus U is not LSE.*[10]

[10] The expected utility function is not LSE and therefore it is not SM either, which from Lemma 13 can only be true if the state utility function is not SM in the Euclidean (sub)lattice. This can be verified in this example.

Thus, the LSE property of the state utility is necessary for the LSE property of the expected utility in the product FOSD lattice, but not sufficient. More is required in order for the expected utility to be LSE (or SSE). It is well known from univariate models of preference over risk that a distribution function First Order Stochastic Dominates another if and only if it is preferred by every expected utility maximiser with increasing utility. Introducing a second element to the utility function does not alter this result if the utility function is also increasing in this second element. This yields a way of linking the lattice theoretic properties of the state and expected utility functions with the standard properties of desirability in the Euclidean lattice. This link is established in Proposition 15. While an increasing function is SSE, the reverse is not true. Moreover, while the order theoretic concept of increasing applies to comparable elements, the lattice theoretic properties apply to non-comparable elements and in particular to distribution functions that are not ordered by FOSD.

Proposition 15 *(i) If the state utility function $u(x,y)$ is non-decreasing (in x and y) and LSE in $\left(\mathbb{R}^2_+, \leq\right)$ then the expected utility function $U(x,\widetilde{y})$ is LSE in $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{(\varepsilon\times FOSD)}\right)$ and (ii) if $u(x,y)$ is increasing in x and y then $U(x,\widetilde{y})$ is SSE in $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{(\varepsilon\times FOSD)}\right)$.*

Proof. Suppose that $X = (x,\widetilde{y})$ and $X' = (x',\widetilde{y}')$ are two incomparable points in $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{(\varepsilon\times FOSD)}\right)$, with $x' \geq x$ and $F_{y'} \ngeq_{FOSD} F_y$. Denote $F_y \vee F_{y'}$ by $F^\vee$ and $F_y \wedge F_{y'}$ by $F^\wedge$. There are two possible cases for such X, X': (1) either $x' \geq x$ and $F_{y'}, F_y$ incomparable, or (2) $x' > x$ and $F_{y'} <_{FOSD} F_y$. In case (1) $F^\vee >_{FOSD} F_y, F_{y'}$, and $F^\wedge <_{FOSD} F_y, F_{y'}$ and in case (2) $F^\vee = F_y$ and $F^\wedge = F_{y'}$. In both cases $F^\vee >_{FOSD} F_{y'}$ and $F^\vee \geq_{FOSD} F_y$.

Consider first part (ii): Suppose that u is increasing. $U(x,\widetilde{y})$ is SSE if:

$$\int u(x',s)dF_{y'} \geq \int u(x,s)dF^\wedge \Rightarrow \int u(x',s)dF^\vee > \int u(x,s)dF_y \tag{9}$$

and

$$\int u(x,s)dF_y \geq \int u(x,s)dF^\wedge \Rightarrow \int u(x',s)dF^\vee > \int u(x',s)dF_{y'} \tag{10}$$

Beginning with the first SSE condition, the second inequality in (9) necessarily holds: In case (1) $u(x',s) \geq u(x,s)$ all s, by u increasing in x (strict inequality if $x' > x$) and $F^\vee >_{FOSD} F_y$ and u increasing in y imply $\int u(x,s)dF^\vee > \int u(x,s)dF_y$. In case (2) $u(x',s) > u(x,s)$ for all s by u increasing in x. In the second LSE condition (10), the second inequality holds since $F^\vee >_{FOSD} F_{y'}$ and u increasing in y (the first inequalities in the two conditions also hold as strict inequalities). Thus $U(x,\widetilde{y})$ is SSE.

Consider part (i) Suppose u is non-decreasing and LSE. $U(x,\widetilde{y})$ is LSE if:

$$\int u(x',s)dF_{y'} \geq (>) \int u(x,s)dF^\wedge \Rightarrow \int u(x',s)dF^\vee \geq (>) \int u(x,s)dF_y \tag{11}$$

and

$$\int u(x,s)dF_y \geq (>) \int u(x,s)dF^{\wedge} \Rightarrow \int u(x',s)dF^{\vee} \geq (>) \int u(x',s)dF_{y'} \quad (12)$$

Since u is non-decreasing then all inequalities in the two conditions are satisfied as weak inequalities. So what remains to be verified is that a strict first inequality implies a strict second inequality. Consider the first LSE condition, (11). In case (2) it becomes

$\int u(x',s)dF_{y'} \geq (>) \int u(x,s)dF_{y'} \Rightarrow \int u(x',s)dF_y \geq (>) \int u(x,s)dF_y$.

If the implication is an equality, $\int u(x',s)dF_y = \int u(x,s)dF_y$, it must be that $u(x',s) = u(x,s)$ (since $u(x',s) \geq u(x,s)$ all s). Therefore $\int u(x',s)dF_{y'} \geq (>) \int u(x,s)dF_{y'}$ must, in fact, be an equality. Thus the condition is satisfied.

Consider case (1). Suppose that the first inequality in 11 is strict, i.e. $\int u(x',s)dF_{y'} > \int u(x,s)dF^{\wedge}$ but the second is satisfied with equality $\int u(x',s)dF^{\vee} = \int u(x,s)dF_y$. Hence, $\int u(x',s)d\left[F^{\vee} - F_{y'}\right] < \int u(x,s)d\left[F_y - F^{\wedge}\right]$. This implies $\int u(x',s)d\left[F^{\vee} - F_{y'}\right] < \int u(x,s)d\left[F^{\vee} - F_{y'}\right]$ since $F^{\vee} + F^{\wedge} = F_y + F_{y'}$ which in turn can be written as

$\int \left[u(x',s) - u(x,s)\right] dF^{\vee} < \int \left[u(x',s) - u(x,s)\right] dF_{y'}$.

By the LSE property of u, $u(x',s) - u(x,s)$ is non-decreasing in s and therefore $\int \left[u(x',s) - u(x,s)\right] dF^{\vee} \geq \int \left[u(x',s) - u(x,s)\right] dF_{y'}$, a contradiction. Thus condition (11) is satisfied.

Next, consider the second LSE condition (12). In case (2) it becomes $\int u(x,s)dF_y \geq (>) \int u(x,s)dF_{y'} \Rightarrow \int u(x',s)dF_y \geq (>) \int u(x',s)dF_{y'}$. Suppose that the first inequality is strict and the second is an equality. Hence it must be that $\int \left[u(x',s) - u(x,s)\right] dF_y < \int \left[u(x',s) - u(x,s)\right] dF_{y'}$. Again $u(x',s) - u(x,s)$ is non-decreasing in s by the LSE property of u. Therefore $\int \left[u(x',s) - u(x,s)\right] dF_y \geq \int \left[u(x',s) - u(x,s)\right] dF_{y'}$, a contradiction. Hence (12) is satisfied.

Consider condition (12) under case (1). Suppose that $\int u(x',s)dF^{\vee} = \int u(x',s)dF_{y'}$, or $\int u(x',s)d\left[F^{\vee} - F_{y'}\right] = 0$. Using $F^{\vee} + F^{\wedge} = F_y + F_{y'}$, then $\int u(x',s)d\left[F_y - F^{\wedge}\right] = 0$. If the first inequality is strict, $\int u(x,s)dF_y > \int u(x,s)dF^{\wedge}$. This implies $\int \left[u(x',s) - u(x,s)\right] dF_y < \int \left[u(x',s) - u(x,s)\right] dF^{\wedge}$. Again $u(x',s) - u(x,s)$ is non-decreasing in s by the LSE property of u. Therefore

$\int \left[u(x',s) - u(x,s)\right] dF_y \geq \int \left[u(x',s) - u(x,s)\right] dF^{\wedge}$, a contradiction.

Hence (12) is satisfied, and U is LSE in the product FOSD lattice. ■

The following example shows that when the LSE property of the state utility function in the Euclidean lattice fails, the non-decreasing property may not be sufficient for the LSE property of the expected utility function in the product FOSD lattice:

Example 16 *Suppose that* u *is defined on* $\{0,1\} \times \{0,1\}$ *as* $u(0,0) = 0$, $u(0,1) = u(1,0) = u(1,1) = 1$. *Therefore,* u *is non-decreasing but not LSE. Take* $\widetilde{y} = \{\pi, 1-\pi; 0, 1\}$, *and* $\widetilde{y}' = \{\pi', 1-\pi'; 0, 1\}$. *Suppose* $\pi < \pi'$. *Then* $X = (0, \widetilde{y})$ *and* $X' = (1, \widetilde{y}')$ *are two incomparable points in* $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{(\varepsilon \times FOSD)}\right)$ *with* $x' \geq x$ *and* $F_{y'} <_{FOSD} F_y$. *Their meet and join are* $X \vee X' = (1, \widetilde{y})$ *and* $X \wedge X' = (0, \widetilde{y}')$ *respectively. It follows that* $U(X') = 1 = U(X \vee X')$ *but* $U(X) = (1-\pi) > (1-\pi') = U(X \wedge X')$, *so* U *is not LSE.*

The results of this section give sufficient conditions for the expected utility function to be LSE in the product FOSD lattice. These are that the state utility function is SM in the Euclidean lattice, or that the state utility function is increasing in the Euclidean lattice, or that it is non-decreasing and LSE in the Euclidean lattice (the third condition implies the second). These results can be used in conjunction with standard analysis techniques when the goods are perfect substitutes in the state utility function. However, they also indicate very strongly that in terms of applying the main lattice programming comparative statics methods, the product FOSD lattice is not very useful. Desirability properties alone in the Euclidean lattice are not sufficient for much comparative statics, and therefore it should not be expected that lattice properties that are derived from them are much more useful.

3.2 The product order and lattice with SOSD

Before we proceed in the next section to construct orders and lattices that can be used in comparative statics analysis, we consider here the case of Second Order Stochastic Dominance (SOSD) in a product order and lattice. As with FOSD, SOSD partially orders the set of distribution functions, $\mathcal{F}_y$, in a way that is compatible with "increase" in risk. We know that in the univariate case a distribution Second Order Stochastic dominates another if and only if it is preferred by all increasing risk averse preferences. This result helps us derive the relation between the lattice theoretic property of the expected utility function and properties of the state utility in the bivariate context.

Let $\leq_{SOSD}$ be defined by $F_y \leq_{SOSD} F_{y'}$ if and only if $\int^s F_y(t)dt \geq \int^s F_{y'}(t)dt$ for all $s \in \mathbb{R}_+$, $F_y, F_{y'} \in \mathcal{F}_y$. The set of distribution functions is a lattice with the SOSD partial order:

Lemma 17 $(\mathcal{F}_y, \leq_{SOSD})$ *is a lattice.*
Proof. *First,* $(\mathcal{F}_y, \leq_{SOSD})$ *is a poset.* $\leq_{SOSD}$ *is a partial order, since* $F_y \leq_{SOSD} F_y$*. Moreover* $F_y \leq_{SOSD} F_{y'}$ *and* $F_y' \leq_{SOSD} F_y$ *imply* $F_y = F_y'$*, and* $F_y \leq_{SOSD} F_{y'}$ *and* $F_{y'} \leq_{SOSD} F_{y''}$ *imply* $F_y \leq_{SOSD} F_{y''}$*. The join and meet of incomparable* $F_y, F_{y'} \in \mathcal{F}_y$ *are given by*

$$F^{\vee}(s) \equiv [F_y \vee F_{y'}](s) = \begin{cases} F_y(s) & \text{if } \int^s F_y \leq \int^s F_{y'} \\ F_{y'}(s) & \text{otherwise} \end{cases} \quad \text{and}$$

$$F^{\wedge}(s) \equiv [F_y \wedge F_y'](s) = \begin{cases} F_y(s) & \text{if } \int^s F_y \geq \int^s F_{y'} \\ F_{y'}(s) & \text{otherwise} \end{cases} .$$

Since $F^{\wedge}, F^{\vee} \in \mathcal{F}_y$*, the poset* $(\mathcal{F}_y, \leq_{SOSD})$ *is a lattice.* ■

Note that $\leq_{SOSD}$ is a coarser ordering of $\mathcal{F}_y$ than $\leq_{FOSD}$, as $F_y \leq_{FOSD} F_{y'} \Rightarrow F_y \leq_{SOSD} F_{y'}$.

The consumption set $\mathbb{R}_+ \times \mathcal{F}_y$ is a lattice when endowed with the product SOSD order, so that $(x, F_y) \leq_{(\varepsilon \times SOSD)} (x', F_{y'})$ iff $x \leq x'$ and $F_y \leq_{SOSD} F_{y'}$.

Lemma 18 $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{(\varepsilon \times SOSD)}\right)$ *is a lattice.*

Proof. Let (x, F_y) and $(x', F_{y'})$ be two points in $\mathbb{R}_+ \times \mathcal{F}_y$. The join and meet are defined as $(x \vee x', F_y \vee_{SOSD} F_{y'})$ and $(x \wedge x', F_y \wedge_{SOSD} F_{y'})$, and both are elements of $\mathbb{R}_+ \times \mathcal{F}_y$. ■

Properties of the state utility function u may again be linked to the lattice properties of the expected utility function U. In the case of SOSD, we can use the fact from univariate analysis that for u concave, a Second Order Stochastic dominant lottery is always preferred. One can thus establish an analog to Proposition 15.

Proposition 19 *If $u(x, y)$ is increasing on $\mathbb{R}_+^2$ and strictly concave in y, then $U(x, \widetilde{y})$ is SSE in $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{(\varepsilon \times SOSD)}\right)$.*

Proof. Suppose that $X = (x, \widetilde{y})$ and $X' = (x', \widetilde{y}')$ are two incomparable points in $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{(\varepsilon \times SOSD)}\right)$, with $x' \geq x$ and therefore $F_{y'} \ngeq_{SOSD} F_y$. $U(x, \widetilde{y})$ is SSE if it satisfies conditions (9) and (10). Suppose that u is increasing. Beginning with the first SSE condition (9), since X and X' are incomparable, either: (i) $\widetilde{y}$ and $\widetilde{y}'$ are incomparable by SOSD so $F^{\vee} >_{SOSD} F_y$, in which case

$\int u(x', s) dF^{\vee} \geq \int u(x, s) dF^{\vee} > \int u(x, s) dF_y$

since u is strictly concave,

or (ii) $F^{\vee} = F_y >_{SOSD} F_{y'}$, and it follows that $x' > x$ since X and X' are incomparable, and therefore $u(x', y) > u(x, y)$ for all y, so $\int u(x', s) dF^{\vee} > \int u(x, s) dF^{\vee} = \int u(x, s) dF_y$.

Regarding the second LSE condition (10), from $F^{\vee} >_{SOSD} F_{y'}$ the second inequality holds since u is strictly concave. ■

Proposition 19 requires strict inequality assumptions on the state utility function in order to establish that it is SSE. Being increasing rules out, for example, Leontieff-type preferences, whereas strict concavity in y rules out risk-neutrality. One might wish to weaken these assumptions in order to obtain sufficient conditions for U to be LSE. This is not as simple in the case of SOSD as in the case of FOSD. As the Example 20 illustrates, conditions analogous to those of Proposition 15 fail to ensure that U is LSE:

Example 20 *Suppose that u is defined on $\{0, 1\} \times \{0, 1, 2\}$. The values of u at $(0,0), (0,1), (0,2)$ are $1, 3, 4$ respectively, and at $(1,0), (1,1), (1,2)$, are $0, 3, 6$, respectively. Then, u is LSE, non-decreasing, and (consistent with) concave in y. Define two lotteries, $\widetilde{y}' = \{0.5, 0.5; 0, 2\}$, and $\widetilde{y} = \{1; 1\}$ (degenerate distribution at 1). Then, $F_{y'} <_{SOSD} F_y$. Take $X = (0, \widetilde{y})$ and $X' = (1, \widetilde{y}')$ as two incomparable points. Their meet and join are $X \vee X' = (1, \widetilde{y})$ and $X \wedge X' = (0, \widetilde{y}')$. Then, $U(X) = 3 = U(X \vee X')$, but $U(X') = 0.5 \times 0 + 0.5 \times 6$ $= 3 > 2.5 = 0.5 \times 1 + 0.5 \times 4 = U(X \wedge X')$ and therefore U is not LSE.*

Both product lattices $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{(\varepsilon \times FOSD)}\right)$ and $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{(\varepsilon \times SOSD)}\right)$ provide insight into the lattice properties of preferences that are used to characterize comparative statics. However, much like the Euclidean lattice $\left(\mathbb{R}_+^2, \leq_{\varepsilon}\right)$ in the certainty case, they are not well adapted to choice problems. This is manifested in the same way as in the certainty case. Namely the relevant economic constraint sets are not strong set ordered and thus the lattice programming comparative statics theorem fails to apply. Since the LSE properties of the expected utility function in these lattices can be established from standard conditions on the state function in the Euclidean lattice, this should come as no surprise.

The $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{(\varepsilon \times FOSD)}\right)$ and $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{(\varepsilon \times SOSD)}\right)$ lattices are well-adapted to certain kinds of increasing constraint sets, but not to those that arise in individual choice problems. Specifically, suppose that individual has income I and the price of the sure good, x, is p_x, and while the price of the risky good, $\widetilde{y}$, is given by a price function $p_y(\widetilde{y})$, with $p_y : \mathcal{F}_y \to \mathbb{R}_+$. The consumer then has the budget constraint $B(I) = \{(x, \widetilde{y}\,| p_x x + p(\widetilde{y}) = I)\}$. Then, the budget sets for different levels of income are not ranked with respect to the strong set order over the lattice $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{(\varepsilon \times FOSD)}\right)$. Suppose that $I' > I$, then $B(I') \ngeq_a B(I)$. In order to see this let $X = (x, \widetilde{y}) \in B(I)$ and $X' = (x', \widetilde{y}') \in B(I')$ be two incomparable bundles, with $x > x'$ and $F_{y'} >_{FOSD} F_y$. Then, $X \vee X' = (x, \widetilde{y}')$, and $p_x(x \vee x') + p_y(\widetilde{y} \vee \widetilde{y}') = p_x x + p(\widetilde{y}') > I'$, that is $X \vee X' \notin B(I')$.

This problem is similar to the one that arises in the certainty case. The solution to this, by analogy with the certainty case, is to use value lattices that can rank budget sets. This allows Theorem 4 to be usefully applied. This generalisation of value orders to the uncertainty case is the object of the next section.

4 Choice over risk with two goods

We now address the main problem of this paper, namely the characterization of income effects in consumer choice with a risky good. That is, the consumption set remains $\mathbb{R}_+ \times \mathcal{F}_y$, and the decision problem is:[11]

$$\begin{array}{cc} \max\limits_{(x,\widetilde{y}) \in \Re_+ \times \mathcal{F}_y} & U(x, \widetilde{y}) = Eu(x, \widetilde{y}) \\ s.t. & p_x x + p(\widetilde{y}) \leq I \end{array} . \tag{13}$$

We assume that the state utility function, u, is increasing in both elements, so that both goods are desirable. Therefore, from Proposition 15, the expected utility function is SSE in $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{(\varepsilon \times FOSD)}\right)$ and also a lottery that first order stochastically dominates another, is preferred.

Observe that the price of the risky good is no longer linear.[12] Instead we posit a general price function $p_y(\widetilde{y})$, with $p_y : \mathcal{F}_y \to \mathbb{R}_+$. We impose various restrictions on this price function in order to accommodate the construction of the consumption value lattice.

It is worth repeating that this is a much richer framework in which choice is made than in the standard choice under uncertainty framework. Not only is

[11] We allow free disposal of income in this problem. This may not be appropriate in problems of choice under uncertainty. Our analysis remains valid if instead of a weak inequality, the budget constraint is an equality constraint.

[12] With different lotteries/ distribution functions feasible, linear pricing does not of itself define a price structure.

there the central issue of imperfect substitutability with a second good in preference, but also with respect to the risky good, the consumption set encompasses all different lotteries. In particular, different quantities of the same lottery are subsumed in this set. Thus, this formulation also encompassses the standard problem where a consumer may choose different quantities of one underlying lottery, or between two or a finite number of lotteries. This generality comes at a cost. It is more difficult to derive strong results in this setting than in a restricted consumption set (and we impose restrictions in order to apply the main comparative statics theorem). We apply our methodology developed here in restricted settings, such as the choice of quantity of one lottery in different papers. However, it is important to lay out the framework in as much generality as possible here. Other extensions such as two risky goods entering separately preferencces, build on this basic model.

It is not straightfoward to give meaning to the notion that the choice of the lottery is *increasing* in income. We follow the intuition from the previous section and seek conditions for choices that are comparable with respect to FOSD. This captures the idea that as the consumer gets wealthier and is able to afford a *better* lottery he chooses to do so. The case of SOSD can proceed along similar lines. A different idea is to focus not on the comparability of the risk chosen, but on the expenditure on the risky good directly. This is attractive because it can be done with the same partial order as in the certainty case, the expenditure value order, thus providing a direct link between the two problems. In this section we show what can be accomplished under each scenario in this general setting.

4.1 Choice over risk and the expenditure value order

As a first approach to the comparative statics of choice under uncertainty we use the expenditure value order introduced in the certainty case. In this case the price function on $\mathcal{F}_y$ represents the cost of the chosen lottery. In the certainty case we imposed the restriction that the price function be increasing. This made the price function invertible. We need a similar condition in order to have a one to one mapping between lotteries and prices. Suppose that the price function on $\mathcal{F}_y$ satisfies an antisymmetry property, that is:

$$p(\widetilde{y}) = p(\widetilde{y}') \quad iff \quad \widetilde{y} = \widetilde{y}' \tag{A1}$$

This property is strong in economic terms, insofar as no two distinct lotteries have identical prices, although a case can be made that if two lotteries are distringuishable, and if they cannot constitute perfect substitutes, their market value should be different. In technical terms, it means that the price function $p_y : \mathcal{F}_y \to \mathbb{R}_+$ is a one-to-one function (this is possible since $\mathcal{F}_y$ has the same cardinality as the real line). Under Assumption A1 the expenditure value order on the consumption set $\mathbb{R}_+ \times \mathcal{F}_y$ is defined as follows.

Definition 21 *Consider the consumption set* $\mathbb{R}_+ \times \mathcal{F}_y$ *and let* $(x, \widetilde{y}), (x', \widetilde{y}')$

$\in \mathbb{R}_+ \times \mathcal{F}_y$. *The expenditure value order is defined by*

$$(x, \widetilde{y}) \leq_{ev(p(\widetilde{y}))} (x', \widetilde{y}') \Longleftrightarrow \begin{array}{c} p(\widetilde{y}) \leq p(\widetilde{y}') \\ p_x x + p(\widetilde{y}) \leq p_x x' + p(\widetilde{y}') \end{array} . \tag{14}$$

It is straightforward to establish that the expenditure value order is a partial order and that $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{ev(p(\widetilde{y}))}\right)$ is a lattice:

Lemma 22 *Let $p_y : \mathcal{F}_y \to \mathbb{R}_+$.satisfy A1. Then $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{ev(p(\widetilde{y}))}\right)$ is a lattice, the expenditure value lattice. The join and meet of a pair of incomparable points $X = (x, \widetilde{y})$ and $X' = (x', \widetilde{y}')$ such that, without loss of generality, $p(\widetilde{y}) > p(\widetilde{y}')$ and $p_x x + p(\widetilde{y}) < p_x x' + p(\widetilde{y}')$ are given by:*

$$X \vee X' = \left(x' - \frac{p(\widetilde{y}) - p(\widetilde{y}')}{p_x}, \widetilde{y}\right), \tag{15}$$

$$X \wedge X' = \left(x + \frac{p(\widetilde{y}) - p(\widetilde{y}')}{p_x}, \widetilde{y}'\right). \tag{16}$$

Proof. In order to establish that $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{ev(p(\widetilde{y}))}\right)$ is a poset, observe that A1 ensures that antisymmetry holds. Transitivity and reflexivity are immediate. In order to establish that it is a lattice, consider first the join of an incomparable pair $X = (x, \widetilde{y})$ and $X' = (x', \widetilde{y}')$ with $p(\widetilde{y}) > p(\widetilde{y}')$ and $p_x x + p(\widetilde{y}) < p_x x' + p(\widetilde{y}')$. Let $z = \left(x' - \frac{p(\widetilde{y}) - p(\widetilde{y}')}{p_x}, \widetilde{y}\right)$. Clearly $z \in \mathbb{R}_+ \times \mathcal{F}_y$ since $x' - \frac{p(\widetilde{y}) - p(\widetilde{y}')}{p_x} \geq 0$ iff $p_x x' + p(\widetilde{y}') \geq p(\widetilde{y})$ which is true by assumption. Also z is an upper bound of X, X', and it is indeed the least upper bound. Similarly $w = \left(x + \frac{p(\widetilde{y}) - p(\widetilde{y}')}{p_x}, \widetilde{y}'\right) \in \mathbb{R}_+ \times \mathcal{F}_y$ and it is the greatest lower bound of X, X' ■

Budget sets for the consumer problem are strong set comparable in the expenditure value lattice:

Lemma 23 *Let $B(I) = \{(x, \widetilde{y}) \mid p_x x + p(\widetilde{y}) \leq I\}$.*[13] *Then $B(I) \leq_a B(I')$ in $(\mathbb{R}_+ \times \mathcal{F}_y, \leq_e v(p(\widetilde{y})))$ for $I \leq I'$.*

Proof. Take X and X' as in Lemma 22, with $p_x x + p(\widetilde{y}) \leq I$ and $p_x x + p(\widetilde{y}) < p_x x' + p(\widetilde{y}') \leq I'$. Then, $p_x x^\wedge + p(\widetilde{y}^\wedge) = p_x x + p(\widetilde{y})$ and $p_x x^\vee + p(\widetilde{y}^\vee) = p_x x' + p(\widetilde{y}')$, so $X \wedge X' \in B(I)$ and $X \vee X' \in B(I')$. ■

Therefore, we can apply the main lattice programming Theorem 4 to establish:

[13] The same result remains valid if instead the budget constraint has to be satisfied with equality.

Proposition 24 *Consider the consumer problem as expressed in 13. Let the consumption set be a lattice with the expenditure value order under assumption A1. Then if the expected utility function is LSE in the expenditure value consumption lattice and* $I \leq I'$

$$\arg\max \{Eu(x, \widetilde{y}) \,|(x, \widetilde{y}) \in B(I)\} \leq_a \arg\max \{Eu(x, \widetilde{y}) \,|(x, \widetilde{y}) \in B(I')\} \quad (17)$$

If instead the expected utility function is SSE and the budget constraint is binding at the optimum, then

$$\arg\max \{Eu(x, \widetilde{y}) \,|(x, \widetilde{y}) \in B(I)\} \leq_s \arg\max \{Eu(x, \widetilde{y}) \,|(x, \widetilde{y}) \in B(I')\} \quad (18)$$

in $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{ev(p(\widetilde{y}))}\right)$.

Proof. Immediate application of Theorem 4. For the second part observe that chain-lower-than comparability implies strongly lower than comparability with binding budget constraints. ■

Thus, the SSE property implies that on every optimum expansion path, the expenditure on the risky good increases. The LSE property implies that such optimal expansion paths, where the expenditure on the risky good increases, exist. This is a strong result. However, in order to evaluate the validity of the assumption and the conclusion, it would be useful to have more information on the price structure of lotteries. As is, we do not have more restriction, other than assumption A1. In the following subsection we impose more structure on prices by imposing that a better lottery, in the sense of FOSD, cannot cost less. We seek conditions so that the choice of the risky good is itself increasing in terms of FOSD. If we impose the restriction that a better lottery, in the sense of FOSD, cannot cost less, then Proposition 24 implies that, in the SSE case, at the higher income the consumer will never choose a lottery that is First Order Stochastically dominated, since the lottery that the consumer chooses costs no less. However, an incomparable lottery that costs more may be chosen. This highlights the fact that additional conditions need to be imposed in order to establish that consumption of the risky good increases with income in the sense of FOSD.

4.2 Choice and the First Order Stochastic Value order

We seek a lattice framework for the consumer allocation problem that can both capture the idea of FOSD increases in the lottery choice $\widetilde{y}$, and that, unlike the product FOSD lattice of the last section, ranks budget sets with respect to the strong set order, $\leq_a$. The solution we propose builds on our work in the certainty case, that is to order bundles with respect to both their value, on the one hand, and the comparative static variable – the lottery choice (with respect to FOSD) on the other. As the set of lotteries $\mathcal{F}_y$ may contain degenerate distributions, in fact this order can be viewed as a generalization of the direct value order (Definition 1).

Definition 25 *Consider the consumption set* $\mathbb{R}_+ \times \mathcal{F}_y$ *and let* $(x, \widetilde{y})$, $(x', \widetilde{y}')$ $\in \mathbb{R}_+ \times \mathcal{F}_y$. *The First Order Stochastic Value (FOSV) order is defined by*

$$(x, \widetilde{y}) \leq_{FOSV(\widetilde{y},p)} (x', \widetilde{y}') \Longleftrightarrow \begin{array}{c} \widetilde{y} \leq_{FOSD} \widetilde{y}' \\ p_x x + p(\widetilde{y}) \leq p_x x' + p(\widetilde{y}') \end{array} \tag{19}$$

One difference between the FOSV and the direct value order in the certainty case is that the definition of the FOSV partial order refers to the lottery price function, $p(\widetilde{y})$. In this sense, the FOSV order is similar to the expenditure value order in the certainty case. In order to construct a consumption lattice with this order we must impose more structure on this price function. Suppose that the price function satisfies Assumptions A2 and A3:

$$F_y \leq_{FOSD} (<_{FOSD}) F_{y'} \Longrightarrow p(\widetilde{y}) \leq (<) p(\widetilde{y}'). \tag{A2}$$

Assumption A2 states that a lottery that (strictly) dominates another by $\leq_{FOSD}$ is (strictly) more expensive. In other words, the ordering on prices reflects the ordering on lotteries. This is a reasonable restriction on prices in economic terms, since it is reasonable that a lottery that is preferred by everybody should cost more than a less preferable one.

The second assumption posits that the price of the degenerate lottery that pays nothing with probability one, $y_0 = \{1; 0\}$, is 0. Since the support of every lottery is in the positive orthant and since the good is desirable this ia not a very stringent assumption.

$$p(y_0) = 0. \tag{A3}$$

The two assumptions on the price function are not unduly restrictive. Notice that they do not necessarily imply assumption A1 which was used in the previous subsection in conjunction with the expenditure value order. Assumptions A2 and A3 can be derived from a more primitive pricing structure, fair pricing of lotteries, based on the prices of the underlying outcomes. Let $a : \Re_+ \longrightarrow \Re_+$ be the price function over the possible outcomes. Then the price of a fairly priced lottery $\mathcal{F}_y$ is given by

$$p(\widetilde{y}) = \int a(y)\, dF_y \tag{A4}$$

Thus, $a(0) = 0$ implies Assumption A3 and Assumption A2 is also satisfied, as is shown in Lemma 26. Consider $\widetilde{y}, \widetilde{y}' \in \mathcal{F}_y$ and suppose that these are incomparable with respect to $\leq_{FOSD}$. Their join and meet are given by: $\widetilde{y}^{\vee}$ with $F^{\vee}(s) = \min\{F_y(s), F_{y'}(s)\}$ and $\widetilde{y}^{\wedge}$ with $F^{\wedge}(s) = \max\{F_y(s), F_{y'}(s)\}$ respectively. Letting $A = \{s \in \mathbb{R}_+ \,|\, F_y(s) \leq\ F_y'(s)\}$ and $A^c = \mathbb{R}_+ \setminus A$, the price of the join and the meet are then given respectively by:

$$p(\widetilde{y}^{\vee}) = \int a(s)\, dF^{\vee}(s) = \int^{A} a(s)\, dF_y(s) + \int^{A^c} a(s)\, dF_{y'}(s) \tag{20}$$

$$p\left(\widetilde{y}^{\wedge}\right)=\int a\left(s\right)dF^{\wedge}\left(s\right)=\int^{A^{c}}a\left(s\right)dF_{y}(s)+\int^{A}a\left(s\right)dF_{y'}(s) \tag{21}$$

It then follows directly that:

Lemma 26 *Consider the set of lotteries $\mathcal{F}_y$ and suppose these are fairly priced according to $(A4)$. Then the prices of comparable lotteries satisfy $(A2)$. Furthermore, the prices of the join and meet of incomparable lotteries, $\widetilde{y},\widetilde{y}'$, in the lattice $(\mathcal{F}_y,\ \leq_{FOSD})$ satisfy*

$$p\left(\widetilde{y}^{\wedge}\right)+p\left(\widetilde{y}^{\vee}\right)=p\left(\widetilde{y}\right)+p\left(\widetilde{y}'\right) \tag{22}$$

It also follows that $\left(\mathbb{R}_{+}\times\mathcal{F}_{y},\leq_{FOSV(\widetilde{y},p)}\right)$ is a poset. Moreover, it is a lattice, called the First Order value lattice, under A2 and A3.

Lemma 27 *Let $p_y:\mathcal{F}_y\rightarrow\mathbb{R}_{+}$.satisfy A2 and A3. Then $\left(\mathbb{R}_{+}\times\mathcal{F}_{y},\leq_{FOSV(\widetilde{y},p)}\right)$ is a lattice. Consider incomparable $(x,\widetilde{y})$, $(x',\widetilde{y}')$ such that w.l.o.g. $p_x x+p\left(\widetilde{y}\right)\leq p_x x'+p\left(\widetilde{y}'\right)$. Their join is given by*

$$\left(x,\widetilde{y}\right)\vee\left(x',\widetilde{y}'\right)=\left(x^{\vee},\widetilde{y}^{\vee}\right)\ \text{where}\ x^{\vee}=\max\left\{x'+\frac{p\left(\widetilde{y}'\right)-p\left(\widetilde{y}^{\vee}\right)}{p_x},0\right\} \tag{23}$$

$$\text{and}\quad F^{\vee}\left(s\right)=\min\left\{F_y\left(s\right),F_y'\left(s\right)\right\}$$

and their meet by

$$\left(x,\widetilde{y}\right)\wedge\left(x',\widetilde{y}'\right)=\left(x+\frac{p\left(\widetilde{y}\right)-p\left(\widetilde{y}^{\wedge}\right)}{p_x},\widetilde{y}^{\wedge}\right)\ \text{with}\ F^{\wedge}\left(s\right)=\max\left\{F_y\left(s\right),F_y'\left(s\right)\right\} \tag{24}$$

Proof. Consider incomparable pair $(x,\widetilde{y})$, $(x',\widetilde{y}')$ with $p_x x+p\left(\widetilde{y}\right)\leq p_x x'+p\left(\widetilde{y}'\right)$. Let $z^{\vee}=\left(\max\left\{x'+\frac{p(\widetilde{y}')-p(\widetilde{y}^{\vee})}{p_x},0\right\},\widetilde{y}^{\vee}\right)$ where $\widetilde{y}^{\vee}$ is defined by $F^{\vee}\left(s\right)=\min\left\{F_y\left(s\right),F_y'\left(s\right)\right\}$, i.e. $\widetilde{y}^{\vee}=\widetilde{y}\vee_{FOSD}\widetilde{y}'$. Clearly $\widetilde{y}^{\vee}$ is well defined, $\widetilde{y}^{\vee}\in\mathcal{F}_y$. Also $z^{\vee}\in\mathbb{R}_{+}\times\mathcal{F}_y$, and it is an upper bound of $(x,\widetilde{y})$, $(x',\widetilde{y}')$, since $p_x x^{\vee}+p\left(\widetilde{y}^{\vee}\right)=\max\left\{p_x x'+p\left(\widetilde{y}'\right),p\left(\widetilde{y}^{\vee}\right)\right\}\geq p_x x'+p\left(\widetilde{y}'\right)$. Consider any other upper bound, $z=(x'',\widetilde{y}'')$. It must be that $\widetilde{y}''\geq_{FOSD}\widetilde{y}^{\vee}$ and therefore $p\left(\widetilde{y}''\right)\geq p\left(\widetilde{y}^{\vee}\right)$ by A2. By definition $p_x x''+p\left(\widetilde{y}''\right)\geq p_x x'+p\left(\widetilde{y}'\right)$. Suppose $p_x x''+p\left(\widetilde{y}''\right)<p_x x^{\vee}+p\left(\widetilde{y}^{\vee}\right)$. If $p_x x^{\vee}+p\left(\widetilde{y}^{\vee}\right)=p_x x'+p\left(\widetilde{y}'\right)>p\left(\widetilde{y}^{\vee}\right)$, then this contradicts $p_x x''+p\left(\widetilde{y}''\right)\geq p_x x'+p\left(\widetilde{y}'\right)$. If $p_x x^{\vee}+p\left(\widetilde{y}^{\vee}\right)=p\left(\widetilde{y}^{\vee}\right)>p_x x'+p\left(\widetilde{y}'\right)$, then $p_x x''+p\left(\widetilde{y}''\right)>p\left(\widetilde{y}^{\vee}\right)$ since $p\left(\widetilde{y}''\right)\geq p\left(\widetilde{y}^{\vee}\right)$. Therefore $z>_{FOSV(\widetilde{y},p)}z^{\vee}$, establishing that $z^{\vee}$ is the join of $(x,\widetilde{y})$, $(x',\widetilde{y}')$. The argument for the meet is straightforward and is omitted. ■

The asymmetry between the join and meet in the lattice $\left(\mathbb{R}_{+}\times\mathcal{F}_{y},\leq_{FOSV(\widetilde{y},p)}\right)$ arises from the fact that assumptions A2 and A3 do not impose much restriction on the price of the join, which may be high, making the bundle $\left(x'+\frac{p(\widetilde{y}')-p(\widetilde{y}^{\vee})}{p_x},\widetilde{y}^{\vee}\right)$ unaffordable. In this case the quantity of good x is reduced to the minimum, zero, but even so the valuation of the join is higher than the more expensive original bundle. This causes problems in establishing strong set budget set comparability of budget sets. This problem does not arise if the consumption set is restricted so that in particular the allowable distribution functions form a chain, $\mathcal{F}_y^C$ with respect to FOSD:

Corollary 28 *Consider* $\left(\mathbb{R}_+ \times \mathcal{F}_y^C, \leq_{FOSV(\widetilde{y},p)}\right)$ *, where* $\mathcal{F}_y^C \sqsubset \mathcal{F}_y$ *a chain with respect to FOSD. This is a sublattice of* $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{FOSV(\widetilde{y},p)}\right)$ *. Let* $(x, \widetilde{y}), (x', \widetilde{y}') \in \mathbb{R}_+ \times \mathcal{F}_y^C$ *be two incomparable points with* $p_x x + p(\widetilde{y}) \leq p_x x' + p(\widetilde{y}')$ *and* $\widetilde{y} \geq_{FOSD} \widetilde{y}'$*. Then,*

$$(x, \widetilde{y}) \wedge (x', \widetilde{y}') = \left(x + \frac{p(\widetilde{y}) - p(\widetilde{y}')}{p_x}, \widetilde{y}'\right) \tag{25}$$

$$(x, \widetilde{y}) \vee (x', \widetilde{y}') = \left(x' - \frac{p(\widetilde{y}) - p(\widetilde{y}')}{p_x}, \widetilde{y}\right) \tag{26}$$

Proof. $\left(\mathbb{R}_+ \times \mathcal{F}_y^C, \leq_{FOSV(\widetilde{y},p)}\right)$ is a sublattice of $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{FOSV(\widetilde{y},p)}\right)$ as $x' + \frac{p(\widetilde{y}') - p(\widetilde{y})}{p_x} \geq x \geq 0$ by construction. ■

The FOSV value order has an intriguing relation with the expenditure value order. It is finer than the latter on the whole consumption set. They are equivalent on subsets of the consumption set, $\mathbb{R}_+ \times \mathcal{F}_y^C$, where the distribution functions form a chain with respect to FOSD:

Lemma 29 *Suppose the price function* $p_y : \mathcal{F}_y \to \mathbb{R}_+$*.satisfies A1, A2 and A3. Consider* $X = (x, \widetilde{y})$ *,* $X' = (x', \widetilde{y}') \in \mathbb{R}_+ \times \mathcal{F}_y$*. Then*

$$(x, \widetilde{y}) \leq_{FOSV(\widetilde{y},p)} (x', \widetilde{y}') \Longrightarrow (x, \widetilde{y}) \leq_{ev(p(\widetilde{y}))} (x', \widetilde{y}')$$

The join and meet in the two corresponding lattices satisfy:

$$X \vee_{ev(p(\widetilde{y}))} X' \leq_{FOSV(\widetilde{y},p)} X \vee_{FOSV(\widetilde{y},p)} X' \tag{27}$$

$$\text{and} \quad X \wedge_{FOSV(\widetilde{y},p)} X' \leq_{FOSV(\widetilde{y},p)} X \wedge_{ev(p(\widetilde{y}))} X' \tag{28}$$

If $X, X' \in \mathbb{R}_+ \times \mathcal{F}_y^c$ *then* $(x, \widetilde{y}) \leq_{ev(p(\widetilde{y}))} (x', \widetilde{y}') \Rightarrow (x, \widetilde{y}) \leq_{FOSV(\widetilde{y},p)} (x', \widetilde{y}')$ *and the two partial orders are equivalent.*

Proof. Since both partial orders are value orders, they both satisfy the same valuation condition. Therefore in order to establish that the FOSV order is finer than the expenditure value order we need to establish that $\widetilde{y} \leq_{FOSD} \widetilde{y}'$ implies $p(\widetilde{y}) \leq p(\widetilde{y}')$. This follows from A2. Note that the reverse is not true since the two lotteries may be incomparable. This is not possible on subsets of the consumption set, $\mathbb{R}_+ \times \mathcal{F}_y^C$ since then $p(\widetilde{y}) \leq p(\widetilde{y}')$ implies $\widetilde{y} \leq_{FOSD} \widetilde{y}'$, and therefore on such subsets the two partial orders are equivalent.

Suppose X, X' are incomparable in $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{ev(p(\widetilde{y}))}\right)$ with $p(\widetilde{y}) > p(\widetilde{y}')$ and $p_x x + p(\widetilde{y}) < p_x x' + p(\widetilde{y}')$. They are also incomparable in $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{FOSV(\widetilde{y},p)}\right)$. Their join in the two lattices is given by

$X \vee_{ev(p(\widetilde{y}))} X' = \left(x' - \frac{p(\widetilde{y}) - p(\widetilde{y}')}{p_x}, \widetilde{y} \right)$ and $X \vee_{FOSV(\widetilde{y},p)} X' = (x^{\vee}, \widetilde{y}^{\vee})$ where $x^{\vee} = \max \left\{ x' + \frac{p(\widetilde{y}') - p(\widetilde{y}^{\vee})}{p_x}, 0 \right\}$.

$p_x \left(x' - \frac{p(\widetilde{y}) - p(\widetilde{y}')}{p_x} \right) + p(\widetilde{y}) = p_x x' + p(\widetilde{y}') \leq p_x x^{\vee} + p(\widetilde{y}^{\vee})$ and $p(\widetilde{y}) \leq p(\widetilde{y}^{\vee})$. Hence $X \vee_{ev(p(\widetilde{y}))} X' \leq_{FOSV(\widetilde{y},p)} X \vee_{FOSV(\widetilde{y},p)} X'$.

Their meet in the two lattices is given by $X \wedge_{ev(p(\widetilde{y}))} X' = \left(x + \frac{p(\widetilde{y}) - p(\widetilde{y}')}{p_x}, \widetilde{y}' \right)$ and $X \wedge_{FOSV(\widetilde{y},p)} X' = \left(x + \frac{p(\widetilde{y}) - p(\widetilde{y}^{\wedge})}{p_x}, \widetilde{y}^{\wedge} \right)$ with the same valuation and with $\widetilde{y}^{\wedge} \leq_{FOSD} \widetilde{y}'$. Hence $X \wedge_{FOSV(\widetilde{y},p)} X'$ $\leq_{FOSV(\widetilde{y},p)} X \wedge_{ev(p(\widetilde{y}))} X'$.

Suppose next that X, X' are comparable in $(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{ev(p(\widetilde{y}))})$ with $p_x x + p(\widetilde{y}) \leq p_x x' + p(\widetilde{y}')$ and $p(\widetilde{y}) \leq p(\widetilde{y}')$, but incomparable in $(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{FOSV(\widetilde{y},p)})$. Therefore $\widetilde{y}$, $\widetilde{y}'$ are incomparable w.r.t. $\leq_{FOSD}$. Again $p_x x' + p(\widetilde{y}') \leq p_x x^{\vee} + p(\widetilde{y}^{\vee})$ and $\widetilde{y}' <_{FOSD} \widetilde{y}^{\vee}$. Hence $X \leq_{ev(p(\widetilde{y}))} X' \leq_{FOSV(\widetilde{y},p)} X \vee_{FOSV(\widetilde{y},p)} X'$. The argument for the meets is analogous and is omitted. ■

Even though the two partial orders can be ordered by refinement, and the joins and meets of incomparable pairs can be compared as in lemma 29, it is not the case that lattice theoretic properties of functions in the two lattices can be similarly compared, or derived from one another. They are different lattices and they have different properties. Here we proceed with $(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{FOSV(\widetilde{y},p)})$. One would like to reason as in the certainty case. Namely, to establish that this lattice ranks budget sets, so that Theorem 4 can be invoked and thus to conclude that the choice of $\widetilde{y}$ is "normal", that is non-decreasing in income with respect to $\leq_{FOSD}$, whenever $U(x, \widetilde{y})$ is SSE (LSE). However, this logic does not exactly apply when the consumption set is $\mathbb{R}_+ \times \mathcal{F}_y$, because budget sets may not be ranked by the induced strong set order, $\leq_a$.

Example 30 *Let $X = (0, \widetilde{y})$ and $X = (0, \widetilde{y}')$ be two incomparable points, with $p(\widetilde{y}') > p(\widetilde{y})$, and take $I = p(\widetilde{y})$ and $I' = p(\widetilde{y}')$. Then, $B(I') \not\geq_a B(I)$. We have, $X \in B(I)$ and $X' \in B(I')$, and $X \wedge X' = \left(\frac{p(\widetilde{y}) - p(\widetilde{y}^{\wedge})}{p_x}, \widetilde{y}^{\wedge} \right) \in B(I)$ (note that $X \wedge X' \in \mathbb{R}_+ \times \mathcal{F}_y$ since $p(\widetilde{y}) > p(\widetilde{y}^{\wedge})$). However, $X \vee X' = (0, \widetilde{y}^{\vee}) \notin B(I')$ as $\widetilde{y}^{\vee} >_{FOSD} \widetilde{y}, \widetilde{y}'$, $p(\widetilde{y}^{\vee}) > p(\widetilde{y}') = I'$.*

This problem is not resolved if fair pricing is assumed either, as the following example shows:

Example 31 *Suppose that $p_x = 1$, and the price function on lotteries is $p(\widetilde{y}) = E\widetilde{y}$. Then, for example, $B(2) \not\geq_a B(1.5)$. Let $X = (0.5, y)$, where y is the degenerate lottery at 1, and hence $p(y) = 1$. Then, $X \in B(1.5)$. Let $X' = (0, \widetilde{y}')$ where $\widetilde{y} = \{0.5, 0.5; 0, 4\}$ and hence so $p(\widetilde{y}') = 2$. Hence $X' \in B(2)$. The join of X, X' is given by $X \vee X' = (0, \widetilde{y}^{\vee})$ where $\widetilde{y}^{\vee} = \{0.5, 0.5; 1, 4\}$ and $p(\widetilde{y}^{\vee}) = 2.5$. Hence $X \vee X' \notin B(2)$.*

This result should not be too surprising. In effect, there is no a priori guarantee that a "better" (in the sense of FOSD) lottery is always affordable. By allowing all possible distribution functions in the consumption set, we allow the possibility that two affordable lotteries are so different that a lottery that dominates both by FOSD is so demanding that it is itself not affordable, even under fair pricing. In the certainty case, the budget frontier is a chain in the consumption value lattice. This is not so here. Suppose the consumer spends all income on a lottery. If the price function is a bijection, then there is a unique lottery that satisfies the budget constraint, say $\widetilde{y}^M$. Any other feasible lottery costs less. Therefore, it is either incomparable or dominated with respect to FOSD by this lottery. Hence $\widetilde{y}^M$ is a maximal element of the budget set with respect to FOSV order, but it is not a maximum. If we assume that the budget set has a maximum, then strong budget set comparability ensues:

Lemma 32 *Suppose the price function $p_y : \mathcal{F}_y \to \mathbb{R}_+$.satisfies A1 and A2. Consider $B(I)$, $B(I')$ with $I' \geq I$ in $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{FOSV(\widetilde{y},p)}\right)$. If $B(I')$ has a maximum with respect to $\leq_{FOSV(\widetilde{y},p)}$, then $B(I) \leq_a B(I')$.*

Proof. Immediate from earlier discussion. $B(I')$ has a maximal element $\widetilde{y}^M$ with $p\left(\widetilde{y}^M\right) = I'$. If this is also a maximum then all feasible lotteries $\widetilde{y} \leq_{FOSD} \widetilde{y}^M$. Therefore, the join of any feasible pair of lotteries is also dominated with respect to FOSD by $\widetilde{y}^M$. Hence its price is no more than I' by $A2$. Hence in 23 $x' + \frac{p(\widetilde{y}')-p(\widetilde{y}^{\vee})}{p_x} \geq 0$. Therefore, the value of the join is equal to the maximum of the values of two bundles, and the join is in $B(I')$. ■

With budget set comparability thus established it is immediate to apply Theorem 4. However, the main assumption in Lemma 32, the existence of a maximum, is not an assumption on primitives, and as such it is not satisfactory. A satisfactory resolution would be to impose restrictions on the primitives of the model (which would in effect implement this assumption). The specified restrictions on pricing of themselves do not resolve the issue. Thus, we may proceed in two different ways. One would be to carry out weaker comparative statics analysis, with weaker feasible set comparability than strong set comparability. This approach was suggested in Antoniadou [1], but the results are not particularly appealing. Another way is to restrict different feasible lotteries, and thus their prices, by restricting their domain. Again this can be done in two different ways. One way is to restrict the feasible lotteries on the (unrestricted) domain. The other way is to restrict the outcome space. A straightforward implementation of the first approach is when feasible lotteries form a chain. Namely, consider the lattice $\left(\mathbb{R}_+ \times \mathcal{F}_y^C, \leq_{FOSV(\widetilde{y},p)}\right)$. As the lotteries in this lattice form a chain, $\widetilde{y}^{\vee} \in \{\widetilde{y}, \widetilde{y}'\}$, so that $p\left(\widetilde{y}^{\vee}\right) = p\left(\widetilde{y}\right) \vee p\left(\widetilde{y}'\right)$, and the join and meet are given by the expressions (25). In this sublattice, $p \cdot X \vee X' = p \cdot X \vee p \cdot X'$ and $p \cdot X \wedge X' = p \cdot X \wedge p \cdot X'$, so that:

Lemma 33 *In $\left(\mathbb{R}_+ \times \mathcal{F}_y^C, \leq_{FOSV(\widetilde{y},p)}\right)$, $B(I) \leq_a B(I')$ whenever $I' \geq I$.*

Therefore, the standard lattice analysis of comparative statics can be conducted, namely that the choice of the risky good, by Theorem 4, is normal whenever U is LSE. The standard hypothesis of much of the existing literature, where only the quantity of a unique underlying lottery is chosen, is captured in the case where the lottery domain is a chain. However, the restriction of the domain to a chain is more general than merely allowing the quantity of one lottery to vary. These issues are discussed in a companion paper, Antoniadou, Mirman and Ruble [3].

Another way of restricting lotteries is by restricting the outcome space. Suppose that all lotteries have positive support on compact subsets of $\mathbb{R}_+$. Then consider any two lotteries. Under fair pricing their price is bounded above by the price of the upper bound outcome and so is that of their join and therefore, if this is less than income, then the join of the two lotteries is affordable, and strong budget set comparability follows. This is in essence the case where the lottery good is "small" relative to income. An analogous assumption is made in existing literature, by considering arbitrarily small risks to the unique good, income. Here the mathematical requirement is to consider a subset of the consumption value lattice with a top.

Lemma 34 *Suppose the price function $p_y : \mathcal{F}_y \to \mathbb{R}_+$.satisfies A2. Let $\widehat{\mathcal{F}}_y \subset \mathcal{F}_y$ such that $\widetilde{y} \in \widehat{\mathcal{F}}_y$ has support on (a subset of) $[0, \overline{y}]$. Let $p((1; \overline{y})) = \overline{\alpha}$, the price of the degenerate lottery at $\overline{y}$. Let $I, I' \in \mathbb{R}_+$ with $I' \geq \overline{\alpha}$ and consider the sublattice $\left(\mathbb{R}_+ \times \widehat{\mathcal{F}}_y, \leq_{FOSV(\widetilde{y},p)}\right)$. Then $B(I) \leq_a B(I')$.*

Proof. Immediate since the join of any two lotteries is dominated wrt FOSD by $(1; \overline{y})$ and therefore its price is no more than $\overline{\alpha}$. ■

Thus again Theorem 4 can be implemented:

Proposition 35 *Consider the consumer problem as expressed in 13. Let the consumption set be a sublattice $\left(\mathbb{R}_+ \times \widehat{\mathcal{F}}_y, \leq_{FOSV(\widetilde{y},p)}\right)$. Then if the expected utility function is LSE and $I \leq I'$ with $I' \geq \overline{\alpha}$*

$$\arg\max \{Eu(x, \widetilde{y}) \,|(x, \widetilde{y}) \in B(I)\} \leq_a \arg\max \{Eu(x, \widetilde{y}) \,|(x, \widetilde{y}) \in B(I')\} \quad (29)$$

If instead the expected utility function is SSE and the budget constraint is binding at the optimum,[14] *then*

$$\arg\max \{Eu(x, \widetilde{y}) \,|(x, \widetilde{y}) \in B(I)\} \leq_s \arg\max \{Eu(x, \widetilde{y}) \,|(x, \widetilde{y}) \in B(I')\} \quad (30)$$

in $\left(\mathbb{R}_+ \times \widehat{\mathcal{F}}_y, \leq_{FOSV(\widetilde{y},p)}\right)$ and thus in $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{FOSV(\widetilde{y},p)}\right)$.

Proof. Immediate application of Theorem 4, using Lemma 34. ■

The sufficient conditions Propositions 24 and 35 relate to lattice theoretic properties of the expected utility function. We may consider whether these can be related to properties of the state function, as was done in the case of the product lattices. For example, one may ask whether a proposition such as

[14] This is true since we assume that both goods are desirable.

Proposition 13 holds in the FOSV lattice. In fact, this is not the case: in the more general non-Euclidean context, the strong equivalence between properties of the state and expected utility functions may not obtain. A weaker result holds, when the space of lotteries is restricted to be a chain ordered by $\geq_{FOSD}$, which imparts a simpler structure to joins and meets.

Proposition 36 *Consider* $\left(\mathbb{R}_+ \times \mathcal{F}_y^C, \leq_{FOSV(\widetilde{y},p)}\right)$. *If the state utility function, u, is concave in x and SM in $\left(\mathbb{R}_+^2, \leq_\varepsilon\right)$, then the expected utility function, U, is SM (hence, LSE) in* $\left(\mathbb{R}_+ \times \mathcal{F}_y^C, \leq_{FOSV(\widetilde{y},p)}\right)$.

Proof. Let $X = (x, \widetilde{y})$ and $X' = (x', \widetilde{y}')$ be two incomparable points in $\left(\mathbb{R}_+ \times \mathcal{F}_y^C, \leq_{FOSV(\widetilde{y},p)}\right)$ with $x' > x$ and $\widetilde{y} >_{FOSD} \widetilde{y}'$. Then, U is SM if:

$$\int u\left(x,s\right)dF_y + \int u(x',s)dF_{y'}$$
$$\leq \int u\left(x + \frac{p\left(\widetilde{y}\right) - p\left(\widetilde{y}'\right)}{p_x}, s\right) dF_{y'} + \int u\left(x' - \frac{p\left(\widetilde{y}\right) - p\left(\widetilde{y}'\right)}{p_x}, s\right) dF_{y'}.$$

Rearrange this as,

$$\int \left[u(x',s) - u\left(x + \frac{p\left(\widetilde{y}\right) - p\left(\widetilde{y}'\right)}{p_x}, s\right)\right] dF_{y'}$$
$$\leq \int \left[u\left(x' - \frac{p\left(\widetilde{y}\right) - p\left(\widetilde{y}'\right)}{p_x}, s\right) - u(x,s)\right] dF_y.$$

If u is supermodular, then $u(x',s) - u\left(x + \frac{p(\widetilde{y})-p\left(\widetilde{y}'\right)}{p_x}, s\right)$ is nondecreasing in s, and since $\widetilde{y} >_{FOSD} \widetilde{y}'$, $dF_{y'}$ can be replaced by dF_y. It is then sufficient that:

$$u(x',s) - u\left(x + \frac{p\left(\widetilde{y}\right) - p\left(\widetilde{y}'\right)}{p_x}, s\right) \leq u\left(x' - \frac{p\left(\widetilde{y}\right) - p\left(\widetilde{y}'\right)}{p_x}, s\right) - u(x,s),$$

that is, that u be concave as $x + \frac{p(\widetilde{y})-p\left(\widetilde{y}'\right)}{p_x} \leq x'$. ■

Using restricted domains in the implementation of the comparative statics theorem, is further justified by observing that the LSE property on the whole of the FOSV lattice is a very demanding property. In fact there is a broad class of utility functions that are not LSE in $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{FOSV(\widetilde{y},p)}\right)$. Suppose that the state utility function u is such that $u\left(x,y\right) = 0$ if $x = 0$, and positive on the interior of $\mathbb{R}_+^2$, as is the Cobb-Douglas utility function. Then, let X and X' be two incomparable points with $\widetilde{y}, \widetilde{y}'$ $FOSD$ incomparable, $p_x x + p\left(\widetilde{y}\right) < p_x x' + p\left(\widetilde{y}'\right)$, and $x' > 0$. If the incomparable points X, X' and the price function $p\left(\widetilde{y}\right)$ are such that $\max\left\{x' - \frac{p\left(\widetilde{y}^\vee\right)-p\left(\widetilde{y}'\right)}{p_x}, 0\right\} = 0$, then $U\left(X \vee X'\right) = 0 \leq U(X), U(X')$. On the other hand, for increasing u, $U(X') = \int u\left(x',s\right)dF_{y'} > \int u\left(x + \frac{p(\widetilde{y})-p\left(\widetilde{y}^\wedge\right)}{p_x}, s\right) dF^\wedge = U\left(X \wedge X'\right)$ since $x + \frac{p(\widetilde{y})-p\left(\widetilde{y}^\wedge\right)}{p_x} < x'$. Therefore, U is not LSE, regardless of any other properties of the state utility function.

On the other hand, keeping this restriction in mind, it is not difficult to find examples of LSE utility functions on this general lattice, a case in point being the perfect substitutes, linear case:

Example 37 *Suppose that the price function satisfies* $p(\widetilde{y}) = p_x E\widetilde{y}$,[15] *and that* $u(x,y) = x + y$. *Taking* $X = (x, \widetilde{y})$ *and* $X' = (x', \widetilde{y}')$, $U(X) = x + E\widetilde{y}$ *and* $U(X') = x' + E\widetilde{y}'$. *Suppose, without loss of generality, that* $p_x x' + p(\widetilde{y}') \geq p_x x + p(\widetilde{y})$ *and* $\widetilde{y}' \ngeq_{FOSD} \widetilde{y}$. *Hence* $X \wedge X' = \left(x + \frac{p(\widetilde{y}) - p(\widetilde{y}^{\wedge})}{p_x}, \widetilde{y}^{\wedge}\right)$ *so* $U(X \wedge X') = x + \frac{p(\widetilde{y}) - p(\widetilde{y}^{\wedge})}{p_x} + E\widetilde{y}^{\wedge} = x + E\widetilde{y} = U(X)$, *and* $X \vee X' = \left(\max\left\{x' - \frac{p(\widetilde{y}^{\vee}) - p(\widetilde{y}')}{p_x}, 0\right\}, \widetilde{y}^{\vee}\right)$ *so* $U(X \vee X') \geq x' - \frac{p(\widetilde{y}^{\vee}) - p(\widetilde{y}')}{p_x} + E\widetilde{y}^{\vee} = x' + E\widetilde{y}' = U(X')$, *so* $U(x,y)$ *is LSE in* $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{FOSV(\widetilde{y},p)}\right)$.

Finally consider the following example which illustrates the importance of the restriction to $\mathcal{F}_y^C$ in Proposition 36. Even when $x' - \frac{p(\widetilde{y}^{\vee}) - p(\widetilde{y}')}{p_x} \geq 0$ so that the join has the "conventional" expression $\left(x' - \frac{p(\widetilde{y}^{\vee}) - p(\widetilde{y}')}{p_x}, \widetilde{y}^{\vee}\right)$, the fact that the meet and join lotteries generally differ from $\widetilde{y}$ and $\widetilde{y}'$ invalidates the link between the state utility function and the expected utility.

Example 38 *Suppose that* $u(x,y) = \sqrt{xy}$, *which is concave and SM in* $\left(\mathbb{R}^2_+, \leq_{\varepsilon}\right)$, *and SM in* $\left(\mathbb{R}^2_+, \leq_{dv(p,y)}\right)$. *Let* $\widetilde{y} = (0.5, 0.5; 1, 4)$ *and* $\widetilde{y}' = (0.5, 0.5; 2, 3)$, *so* $\widetilde{y}^{\vee} = (0.5, 0.5; 2, 4)$ *and* $\widetilde{y}^{\wedge} = (0.5, 0.5; 1, 3)$. *Moreover, assume that prices are such that* $\frac{p(\widetilde{y}^{\vee}) - p(\widetilde{y}')}{p_x} = \frac{p(\widetilde{y}) - p(\widetilde{y}^{\wedge})}{p_x} = 4$, *and take* $X = (5, \widetilde{y})$, $X' = (10, \widetilde{y}')$ *so* $X \vee X' = (6, \widetilde{y}^{\vee})$ *and* $X \wedge X' = (9, \widetilde{y}^{\wedge})$. *Then,* $U(X) + U(X') = 8.33$, $U(X \vee X') + U(X \wedge X') = 8.28$, *so although* U *is SM in* $\left(\mathbb{R}_+ \times \mathcal{F}_y^C, \leq_{FOSV(\widetilde{y},p)}\right)$ *by Proposition 36, it is not SM in* $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{FOSV(\widetilde{y},p)}\right)$.

4.3 Second order stochastic dominance

The analysis developed for the case of FOSV lattices translates to the context of other stochastic orders. We sketch the case where lotteries, and therefore choices, are ranked with regard to SOSD. Suppose that the order on prices is consistent with the underlying SOSD order:

$$F_y \leq_{SOSD} (<_{SOSD}) F_{y'} \Longrightarrow p(\widetilde{y}) \leq (<) \, p(\widetilde{y}'). \qquad \text{((A5))}$$

By analogy with the last subsection, we begin by defining a value order:

Definition 39 *Consider the consumption set* $\mathbb{R}_+ \times \mathcal{F}_y$ *and let* $(x, \widetilde{y}), (x', \widetilde{y}') \in \mathbb{R}_+ \times \mathcal{F}_y$. *The second order stochastic value order (SOSV) is defined by*

$$(x, \widetilde{y}) \leq_{SOSV(\widetilde{y},p)} (x', \widetilde{y}') \Longleftrightarrow \begin{array}{c} \widetilde{y} \leq_{SOSD} \widetilde{y}' \\ p_x x + p(\widetilde{y}) \leq p_x x' + p(\widetilde{y}') \end{array} \qquad (31)$$

[15] Notice that this does not satisfy assumption A1, which is however not required here.

The consumption set with the second order value order is a poset and also a lattice, called the Second Order value lattice:

Lemma 40 *Suppose the price function* $p_y : \mathcal{F}_y \to \mathbb{R}_+$*.satisfies (A5) and A3. Then* $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{SOSV(\widetilde{y},p)}\right)$ *is a lattice. Consider two incomparable points* $(x, \widetilde{y})$, $(x', \widetilde{y}')$ *such that w.l.o.g.* $p_x x + p(\widetilde{y}) \leq p_x x' + p(\widetilde{y}')$*. Their join is given by*

$$(x, \widetilde{y}) \vee (x', \widetilde{y}') = (x^\vee, \widetilde{y}^\vee) \ \ where\ x^\vee = \max\left\{x' + \frac{p(\widetilde{y}') - p(\widetilde{y}^\vee)}{p_x}, 0\right\} \quad (32)$$

$$and \ \ F^\vee(s) = \begin{cases} F_y(s) \ \ if \ \ \int^s F_y(t)\,dt \leq \int^s F_y'(t)\,dt \\ F_{y'}(s) \qquad\qquad\qquad otherwise \end{cases}$$

and their meet by

$$(x, \widetilde{y}) \wedge (x', \widetilde{y}') = \left(x + \frac{p(\widetilde{y}) - p(\widetilde{y}^\wedge)}{p_x}, \widetilde{y}^\wedge\right) \quad (33)$$

$$with\ F^\wedge(s) = \begin{cases} F_y(s) \ \ if \ \ \int^s F_y(t)\,dt \geq \int^s F_y'(t)\,dt \\ F_{y'}(s) \qquad\qquad\qquad otherwise \end{cases}$$

Proof. The proof is analogous to the one for the First order value lattice and is therefore omitted. ■

As in the case of the first order value lattice, budget sets are not strong set comparable in $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{SOSV(\widetilde{y},p)}\right)$ unless further restrictions are imposed. Similar conditions to the ones imposed in the First order value lattice can be used.

5 Final remarks

We conclude with some remarks regarding the method introduced in this paper.

First, our results focus on stochastic dominance which is a natural place to begin. Stochastic dominance is a natural ordering on distributions. The same steps (defining a value order so as to obtain a lattice that ranks the constraint sets, then characterizing comparative statics by means of theorem 4) apply in cases of different orderings, such as the Monotone Likelihood Ratio.

Second, the issue of the relation between lattice theoretic properties of the expected utility function and the state utility function requires further investigation. This is more complicated in the value lattices than in the product lattices. The issue of determining conditions for U to be LSE (SSE) in specific applications is explored further in a companion paper.

Finally, we emphasize that $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{FOSV(\widetilde{y},p)}\right)$ is but one possible value lattice, and one may obtain analogous results by appealing to different types of value orders. For example, the radial value order (Definition 7) in the certainty case compares bundles with respect to rays from the origin: for a given value, a bundle is larger if it is has a higher ratio of y to x. An increase with respect to the radial order is consistent with increases in y, which is consistent with normality. In fact a stronger property obtains, namely that an increase with respect to the radial order reflects a larger expenditure share of good y. Moreover, in the certainty case, some functions that are not LSE in the direct value order are LSE in the radial value order. The fact that an array of orders can be usefully applied to a given problem (so long as the constraint sets are ranked in the resulting lattice) enhances the applicability of lattice methods.

One can proceed along the same lines in the stochastic case. That is, it is possible to define an alternative stochastic order to $\leq_{FOSV(\widetilde{y},p)}$ which is based on a comparison by stochastic dominance, not of the distributions $\widetilde{y}$, but of the (stochastic) intensities $\frac{\widetilde{y}}{x}$.

Definition 41 *Suppose the price function satisfies A1, A2 and A3.*[16] *Consider the consumption set* $\mathbb{R}_+ \times \mathcal{F}_y$ *and let* $(x, \widetilde{y})$, $(x', \widetilde{y}') \in \mathbb{R}_+ \times \mathcal{F}_y$. *Then, define* $\leq_F$ *OSRV* $(\widetilde{y}, p)$, *the first order stochastic radial value order as* $(x, \widetilde{y}) \leq_F$ *OSRV* $(\widetilde{y}, p)\,(x', \widetilde{y}')$ *if and only if,*

$$\left\{ \begin{array}{c} x'\widetilde{y} \leq_{FOSD} x\widetilde{y}' \\ p_x x + p(\widetilde{y}) \leq p_x x' + p(\widetilde{y}') \end{array} \right. , \tag{34}$$

We establish that $\leq_{FOSRV(\widetilde{y},p)}$ is indeed a partial order.

Lemma 42 $\leq_{FOSRV(\widetilde{y},p)}$ *is a partial order on* $\mathbb{R}_+ \times \mathcal{F}_y$

Proof. Reflexivity is immediate. To establish antisymmetry, suppose that $(x, \widetilde{y}) \leq_{FOSRV(\widetilde{y},p)} (x', \widetilde{y}')$ and $(x', \widetilde{y}') \leq_{FOSRV(\widetilde{y},p)} (x, \widetilde{y})$. Suppose that $x' > x$. Then from $x'\widetilde{y} \leq_{FOSD} x\widetilde{y}'$, $\widetilde{y} \leq_{FOSD} \frac{x}{x'}\widetilde{y}'$ and therefore $\widetilde{y}' >_{FOSD} \widetilde{y}$. Hence by A2 $p(\widetilde{y}') > p(\widetilde{y})$, so $p_x x' + p(\widetilde{y}') > p_x x + p(\widetilde{y})$, contradicting $(x', \widetilde{y}') \leq_{FOSRV(\widetilde{y},p)} (x, \widetilde{y})$. The case $x' < x$ is the same. Hence $x = x'$. Suppose that $x = x' \neq 0$, then $\widetilde{y} \leq_{FOSD} \widetilde{y}'$ and $\widetilde{y} \geq_{FOSD} \widetilde{y}'$ so $\widetilde{y} = \widetilde{y}'$. If $x = x' = 0$ and therefore $p(\widetilde{y}) = p(\widetilde{y}')$. Then by A1 $\widetilde{y} = \widetilde{y}'$.

In order to establish transitivity suppose that $(x, \widetilde{y}) \leq_{FOSRV(\widetilde{y},p)} (x', \widetilde{y}')$ and $(x', \widetilde{y}') \leq_{FOSRV(\widetilde{y},p)} (x'', \widetilde{y}'')$. It follows that $p_x x + p(\widetilde{y}) \leq p_x x'' + p(\widetilde{y}'')$. Furthermore $x'\widetilde{y} \leq_{FOSD} x\widetilde{y}'$ and $x''\widetilde{y}' \leq_{FOSD} x'\widetilde{y}''$. We need to establish that $x''\widetilde{y} \leq_{FOSD} x\widetilde{y}''$. Suppose that $x'' = 0$. Then, $x''\widetilde{y} \leq_{FOSD} x\widetilde{y}''$ holds. Hence suppose $x'' > 0$. From $(x', \widetilde{y}') \leq_{FOSRV(\widetilde{y},p)} (x'', \widetilde{y}'')$ it must be that $x' > 0$ also, unless $\widetilde{y}'$ is the degenerate lottery at 0, $\widetilde{y}' = \{1; 0\}$. If $x' = 0$ and $\widetilde{y}' = \{1; 0\}$ then $p_x x' + p(\widetilde{y}') = 0$, contradicting $p_x x' + p(\widetilde{y}') > p_x x + p(\widetilde{y})$. Therefore, $x' > 0$. Similarly if $x' > 0$ from $(x, \widetilde{y}) \leq_{FOSRV(\widetilde{y},p)} (x', \widetilde{y}')$ it must also be true that $x > 0$ unless $\widetilde{y} = \{1; 0\}$. If $\widetilde{y} = \{1; 0\}$ then $x''\widetilde{y} \leq_{FOSD} x\widetilde{y}''$ is true, and if $x > 0$

[16] Assumption A1 may be avoided by instead restricting the consumption set to allow only positive consumption of good x, $\mathbb{R}_{++} \times \mathcal{F}_y$, since the problems with establishing that this is a partial order arise only along the x-axis.

then $x''\widetilde{y} \leq_{FOSD} x\widetilde{y}''$ follows directly from $\widetilde{y} \leq_{FOSD} \frac{x}{x'}\widetilde{y}'$ and $\widetilde{y}' \leq_{FOSD} \frac{x'}{x''}\widetilde{y}''$. ■

As with $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{FOSV(\widetilde{y},p)}\right)$, $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{FOSRV(\widetilde{y},p)}\right)$ is a lattice (subject to an additional assumption on prices).[17] Subject to price function and domain restrictions, once the lattice is constructed, the analysis may proceed in analogous fashion to the analysis in $\left(\mathbb{R}_+ \times \mathcal{F}_y, \leq_{FOSV(\widetilde{y},p)}\right)$.

[17] We assume that $p(k\widetilde{y}) = kp(\widetilde{y})$. This states that the lottery price is linear in the "amount" of the risky variable purchased. When this holds, an increase with respect to $\leq_{FOSRV(\widetilde{y},p)}$ corresponds not only to a higher ratio, $\frac{\widetilde{y}}{x}$, but also to an FOSD shift and a higher expenditure share on the risky good.

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