Seasonal Stochastic Volatility and Correlation together with the Samuelson Effect in Commodity Futures Markets

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26-27 November 2015, Lyon
Introduction
Introduction

- Recently, several volatility indices for agricultural markets have been introduced on the Chicago Board Options Exchange (CBOE):
  - in 2011, the Corn Volatility Index (CIV) and Soybean Volatility Index (SIV);
  - and in 2012, the Wheat Volatility Index (WIV).
- Modelling stochastic volatility allows to calibrate to option volatility smiles and skews typically seen in futures option markets.
- A good model should also account for the Samuelson effect that the volatility of nearby futures prices is higher than that of far-away futures prices.
In this presentation, we will:

- Show and discuss some agricultural market data;
- introduce a seasonal stochastic volatility model;
- discuss various seasonality functions;
- show that the model can be fitted very well to the market;
- discuss the pricing of calendar spread options;
- give evidence of the influence of seasonality on prices and implied correlations of such options.
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- Wheat - Volatility Surface
- Corn - Volatility Surface
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**Market Data**

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**Calendar Spread Options**

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A Seasonal Stochastic Volatility Model

The futures price $F(t, T_m)$ at time $t$, $t \leq T_m$, follows the SDE

$$dF(t, T_m) = F(t, T_m) \sum_{j=1}^{n} e^{-\lambda_j (T_m-t)} \sqrt{v_j(t)} dB_j(t),$$

$$F(0, T_m) = F_{m,0}.$$

The processes $v_j$, $j = 1, \ldots, n$, are CIR/Heston square-root stochastic variance processes

$$dv_j(t) = \kappa_j (\theta_j(t) - v_j(t)) dt + \sigma_j \sqrt{v_j(t)} dB_{n+j}(t),$$

$$v_j(0) = v_{j,0}.$$

with time-dependent, deterministic mean-reversion levels $\theta_j(t)$.

We assume

$$\langle dB_j(t), dB_{n+j}(t) \rangle = \rho_j dt,$$

and that otherwise the Brownian motions are independent of each other.
The Seasonality Function

We want to include discontinuous seasonality functions in our framework:

- Let $\mathcal{T} = \{t_i, i = 1, \ldots\}$ be a set of times having only finitely many points in every bounded interval.
- Let $\mathcal{Z} = \{0 \leq t_1 < t_2 < \ldots < t_i < \ldots\}$ be the partition of $\mathbb{R}_0^+$ defined by $\mathcal{T}$.
- Let the seasonality function $\theta : \mathbb{R}_0^+ \to \mathbb{R}^+$ be piecewise continuous with respect to $\mathcal{Z}$.
- Assume that it is bounded from below and above by positive constants $\theta_{\text{min}}$ and $\theta_{\text{max}}$. 
The Seasonality Function

We compare two processes $v$ (seasonal) and $\tilde{v}$ (non-seasonal), which are given by the SDEs

\begin{align*}
    dv(t) &= \kappa (\theta(t) - v(t)) \, dt + \sigma \sqrt{v(t)} \, dB(t), \tag{1} \\
    d\tilde{v}(t) &= \kappa (\theta_{\min} - \tilde{v}(t)) \, dt + \sigma \sqrt{\tilde{v}(t)} \, dB(t), \tag{2}
\end{align*}

with identical parameters $\kappa > 0$, $\sigma > 0$ and initial conditions

\[ 0 < \tilde{v}(0) = \tilde{v}_0 \leq v(0) = v_0. \]
The Seasonality Function

It is well known that (2) has a unique strong solution. The following result describes the solution to (1).

**Proposition 1** Assume that the seasonality function $\theta$ is piecewise continuous w.r.t. the partition $\mathcal{Z}$ of $\mathbb{R}_0^+$, and bounded by positive constants $\theta_{\text{min}}$ and $\theta_{\text{max}}$, i.e. for all $t \geq 0$, $0 < \theta_{\text{min}} \leq \theta(t) \leq \theta_{\text{max}}$. Let the processes $v$ and $\tilde{v}$ be given by (1) and (2), respectively. Then:

1. The process (1) has a unique strong solution with continuous sample paths.
2. $\mathbb{P}[\tilde{v}_t \leq v_t, \forall t \geq 0] = 1$.
3. If the Feller condition $\sigma^2 < 2\kappa \theta_{\text{min}}$ is satisfied for $\theta_{\text{min}}$, then the process $v$ is strictly positive.
Examples of Seasonality Functions

- **Sinusoidal pattern**, with $a, b > 0$ and $t_0 \in [0, 1[$:
  \[
  \theta(t) = a + b \cos (2\pi \omega (t - t_0)).
  \]

- **Exponential-sinusoidal pattern**, with $a, b > 0$ and $t_0 \in [0, 1[$:
  \[
  \theta(t) = a \exp (b \cos (2\pi \omega (t - t_0))).
  \]
  A similar parametric form is used in [1].

- **Sawtooth pattern**, with $a, b > 0$ and $t_0 \in [0, 1[$:
  \[
  \theta(t) = a + b (t - t_0 - \lfloor t - t_0 \rfloor).
  \]
  This is an example of a discontinuous seasonality function.
Examples of Seasonality Functions

- **Triangle pattern**, with $a, b > 0$ and $t_0 \in [0, 1[$:

$$\theta(t) = a + b \left| \frac{1}{2} - (t - t_0 - \lfloor t - t_0 \rfloor) \right|.$$

- **Spiked pattern**, with $a, b > 0$ and $t_0 \in [0, 1[$:

$$\theta(t) = a + b \left( \frac{2}{1 + |\sin(\pi(t - t_0))|} - 1 \right)^2.$$ 

This parametric form is applied to electricity markets in [3] to model the time varying intensity of a jump process.
The Joint Characteristic Function

**Proposition 2** The joint characteristic function $\phi$ at time $T \leq T_1, T_2$ for the log-returns $X_1(T), X_2(T)$ of two futures contracts with maturities $T_1, T_2$ is given by

$$
\phi(u) = \prod_{j=1}^{n} \exp \left( -i \frac{\rho_j}{\sigma_j} f_{j,1}(u, 0) \left( v_j(0) + \kappa_j \hat{\theta}_{j,T} \right) \right) 
\cdot \exp \left( A_j(0, T)v_j(0) + B_j(0, T) \right),
$$

with

$$
f_{j,1}(u, t) = \sum_{k=1}^{2} u_k e^{-\lambda_j(T_k-t)}, \quad f_{j,2}(u, t) = \sum_{k=1}^{2} u_k e^{-2\lambda_j(T_k-t)},
$$

$$
q_j(u, t) = i \rho_j \frac{\kappa_j - \lambda_j}{\sigma_j} f_{j,1}(u, t) - \frac{1}{2} (1 - \rho_j^2) f_{j,1}^2(u, t) - \frac{1}{2} i f_{j,2}(u, t),
$$

$$
\hat{\theta}_{j,T} = \int_0^T \theta_j(t) e^{\lambda_j t} dt.
$$
The functions $A_j : (t, T) \mapsto A_j(t, T)$ and $B_j : (t, T) \mapsto B_j(t, T)$ satisfy the two differential equations

$$\frac{\partial A_j}{\partial t} - \kappa_j A_j + \frac{1}{2} \sigma_j^2 A_j^2 + q_j = 0$$
$$\frac{\partial B_j}{\partial t} + \kappa_j \theta_j(t) A_j = 0$$

with conditions

$$A_j(T, T) = i \frac{\rho_j}{\sigma_j} f_{j,1}(u, T)$$
$$B_j(T, T) = 0$$

Note that only the differential equation for $B_j$, but not the one for $A_j$, depends on the seasonality function $\theta_j$. We give closed-form expressions for the functions $A_j$ in terms of Kummer functions in [5].
The Transform of the Seasonality Function

- The integrals

\[ \hat{\theta}_{j,T} = \int_0^T \theta_j(t)e^{\lambda_j t} dt \]

only depend on the specification of the seasonality functions \( \theta_j \) and the maturity \( T \).
- Therefore, their value can be calculated once and then stored, avoiding recalculations during repeated calls to the characteristic function.
- We show in [6] how to calculate these integrals analytically for the sinusoidal, sawtooth and triangle patterns.
- If \( \theta \) is a constant function, then the joint characteristic function given above is the same as the non-seasonal one given in [5].
Calibration Results with the One-Factor Model

- The One-Factor Model (n = 1)
- Wheat - Futures
- Wheat - Volatility Term-Structure
- Corn - Futures
- Corn - Volatility Term-Structure
- Soybean - Futures
- Soybean - Volatility Term-Structure
- Raw Sugar - Futures
- Raw Sugar - Volatility Term-Structure

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The One-Factor Model (n = 1)

Futures and Variance processes:

\[
\frac{dF(t, T_m)}{F(t, T_m)} = e^{-\lambda(T_m-t)} \sqrt{v(t)} dB_1(t)
\]

\[
dv(t) = \kappa (\theta(t) - v(t)) dt + \sigma \sqrt{v(t)} dB_2(t)
\]

with correlation \( \langle dB_1(t), dB_2(t) \rangle = \rho dt \).

Seasonality function:

\[
\theta(t) = a + b \cos(2\pi \omega(t - t_0))
\]
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- Corn - Volatility
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- Corn - Volatility

Soybean - Futures
- Soybean - Volatility

Raw Sugar - Futures
- Raw Sugar - Volatility

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March 2015

T (years)

0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6

900 910 920 930 940 950 960 970 980 990 1000

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- Wheat - Volatility
- Term Structure
- Corn - Futures
- Corn - Volatility
- Term Structure
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- Soybean - Volatility
- Term Structure
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- Raw Sugar - Volatility
- Term Structure

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- Corn - Futures
- Corn - Volatility Term-Structure
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  - Wheat - Volatility Term-Structure
  - Corn - Futures
  - Corn - Volatility Term-Structure
  - Soybean - Futures
  - Soybean - Volatility Term-Structure
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Calendar spread options are popular options in commodities markets.

Corn, Soybean and Wheat CSOs are traded on the Chicago Mercantile Exchange (CME).

In February 2015 the Minneapolis Grain Exchange (MGX) introduced North American Hard Red Spring Wheat (HRSW) CSOs for trade on the CME.

CSOs in agricultural grain markets are studied in [4] with a joint Heston model for the two underlying futures contracts. The variance process has a constant mean-reversion level, and therefore does not display seasonality.

In the following, we show the effects volatility seasonality can have on CSO prices.
Calendar Spread Options

- Let $K$ (which is allowed to be negative) denote the strike of the option.
- Payoff of a calendar spread call option $CSC$:

$$ (F(T, T_1) - F(T, T_2) - K)^+ = \max(F(T, T_1) - F(T, T_2) - K, 0), $$

- Payoff of a calendar spread put option $CSP$:

$$ (K - (F(T, T_1) - F(T, T_2)))^+ = \max(K - (F(T, T_1) - F(T, T_2)), 0). $$

- We use the 2-factor model together with Caldana and Fusai’s method [2] to price these options.
Pricing CSOs with Caldana and Fusai’s Method

Let $\Phi(u) = \exp \left( i \sum_{k=1}^{2} u_k \ln F(0, T_k) \right) \cdot \phi(u)$ be the joint c.f. of $\ln F(T, T_1)$ and $\ln F(T, T_2)$. The price of a calendar spread option call with maturity $T$ and strike $K$ is given as

$$CSC(K, T, T_1, T_2) = \left( \frac{e^{-\delta k - rT}}{\pi} \int_0^{+\infty} e^{-i\gamma k} \Psi(\gamma; \delta, \alpha) d\gamma \right)^+,$$

where

$$\Psi(\gamma; \delta, \alpha) = \frac{e^{i(\gamma - i\delta) \ln(F(0, -i\alpha))}}{i(\gamma - i\delta)} \cdot \left[ \Phi((\gamma - i\delta) - i, -\alpha(\gamma - i\delta)) - \Phi(\gamma - i\delta, -\alpha(\gamma - i\delta) - i) - K \Phi(\gamma - i\delta, -\alpha(\gamma - i\delta)) \right],$$

$$\alpha = \frac{F(0, T_2)}{F(0, T_2) + K}, \quad k = \ln(F(0, T_2) + K).$$

The parameter $\delta$ controls an exponential decay term.
Comparing No, Weak, and Strong Seasonality

\[ \theta_j(t) = a_j + b_j \cos(2\pi \omega_j(t - t_{0j})), \ j = 1, 2 \]

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<th>Case 3</th>
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<tr>
<td>(\sigma_2)</td>
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<td>(t_{02})</td>
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Calendar Spread Option Prices and Seasonality

<table>
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<th>Case 2: moderate seasonality</th>
<th>Case 3: strong seasonality</th>
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The CSOs in this example are all for six months, i.e. $T_2 - T_1$ is half a year. Note that seasonality can lead to *increasing or decreasing prices*.
Correlation Term-Structure and Seasonality

This effect can also be seen in the change in implied correlations calculated from CSO prices. **Case 1: no seasonality, Case 2: moderate seasonality, Case 3: strong seasonality.**
Literature
Literature


