In this paper, a novel model is proposed to handle temporal dependence, both between claims amounts of each line of business and between claims of the two lines of business. Time varying copula functions with a Generalized Autoregressive Conditional Sinistrality model are proposed to analyse the evolution in time of dependence between two lines as well between claims of each line. Simulation study is performed to highlight the impact on reserves and Solvency Capital Requirement. Results show that a diversification effect could be gained on the amount of reserve when considering temporal dependence structures.

**Keywords:** Claims reserving, Time varying copula models, Generalized Autoregressive Conditional Sinistrality model, Simulation method, Solvency Capital Requirement.

**JEL Classification Codes:** C22, C52, C58, G22
1 Introduction

Forecasting claims reserves and assessing the accurate solvency capital are a crucial issue for actuaries.

Classical approaches assume that claims of different lines of business are independent. However, it is unlikely that claims to different policies are independent. To further illustrate, a car accident can induce claims on both auto and health policies. Also, a fire may spread from one building to another resulting in claims from two or more policyholders. Thus, one event may involve series of claims. In this context, assuming that different policies of lines of business are independent can lead to over or under estimation of the aggregate loss. It has pointed out that methods enabling modelling dependencies of different policies of lines of business are needed, in order to improve the accuracy of the estimation of the total reserves and solvency capital.

Copulas functions have been proposed by several actuarial studies as a relevant tool handling different dependence structures. They were introduced to actuarial mathematics by Frees and Valdez (1998). Since then, a large number of investigations have applied copulas functions to fully capture a wide range of dependence structures amongst different insured risks. Frees and Wang (2006) have used copulas to model dependence between claims, in order to develop credibility predictors of aggregate losses. In addition, Kaishev and Dimitrova (2006) have shown the importance of copulas in reinsurance. They have modelled claim severities by a copula functions. Antonin and Benjamin (2001) also used copulas to assess the amount of the reserve, they provided a model which combined at the same time the theory of copula and the theory of credibility in order to detect better the dependence between the lines of business. Furthermore, Belguise and Levi (2002), Faivre (2002) and Cadoux and Loizeau (2004) have shown that copulas models allow to capture dependence between lines of business and yield an amount of capital higher than when assuming independence. In addition, Bargés et al. (2009) evaluated the capital allocation for the overall portfolio using the TVaR as a measure of risk and a Farlie-Gumbel-Morgenstern (FGM) copula. Besides, Diers et al. (2012) have shown the flexibility of a Bernstein copula to model a several lines of business. Moreover Zhang and Dukic (2013) have introduced a Bayesian multivariate model based on the use of parametric copula to model dependencies between various lines of insurance claims.

More recently, a more relevant technique was introduced in actuarial sciences, which is the copula regression model, where distributions are specified conditional on a set of covariates. Some recent applications include the modeling of insurance claims:
Frees and Valdez (2008), Frees et al. (2009). The prediction of reserves: (Zhao and Zhou (2010), Shi and Frees (2011)). The examination of asymmetric information: (Shi and Valdez (2011), Shi and Valdez (2012)), and the analysis of insurance claims: (Shi and Frees (2010), Shi and Valdez (2012), Krämer et al. (2013)), among others. Despite the popularity of copulas in dependence modelling, their application in time series data is not considered. The majority of approaches proposed in actuarial science do not involve the temporal dependence problem, when a serial dependencies and a time varying dependence structure of claims can be captured.

Recently, a model proposed by Pešta and Okhrin (2014) for time series claims data, introduces a Generalized time series model for conditional mean and variance of one run-off triangle, and elaborates copula approach for dependence modelling within their proposed model. They proved, that their model improves the consistency of parameters estimates and the precision of reserve distribution as well. Therefore it is important to consider the temporal dependence structure of claims, and to model dependence structure with a model that can follow the change in dependence over time. Their paper deals with the temporal dependence structure of one run-off triangle, but time varying dependence structure of several run-off triangles is not considered.

For that, in this paper, a novel model handling time varying dependence structure between two run-off triangles is proposed. Extending prior work that used static copulas, where the dependence structure are presented as a value, this paper used the dynamic copula functions, where dependence structure evolves through time. Therefore, it takes into account the dynamic behaviour of claims. To our knowledge is the first paper to investigate dynamic copulas in insurance.

Also, a time series Generalized linear model namely a Generalized Autoregressive Conditional Sinistrality model is developed, to capture temporal dependence within each triangles. Furthermore, a simulation procedure is provided in order to estimate the aggregate reserves and the total solvency capital.

This paper is organized as follows: Section 2 presents the Autoregressive Conditional Sinistrality Model for marginals modelling. Section 3 provides the time varying copula model. Section 4 reports the empirical results, followed by concluding remarks and some open questions in Section 5.
2 Marginal models

The primary goal of our study is to evaluate reserves and Solvency Capital taking into account the dynamic dependence structure between lines of business. For that, a suitable model for marginals must be identified. In this paper, a Generalized Autoregressive Conditional Sinistrality Model (GACSM) proposed by Araichi et al. (2015) is considered. This model assumes that claims amounts $y_{i,t}$ from the same accident quarter are correlated. A dynamic specification is considered

$$y_{i,t} = \psi_{i,t} \epsilon_{i,t},$$

where $\epsilon_{i,t}$ denote the standardized claims amounts (residuals), that are assumed to belong to the exponential family.

$$\epsilon_{i,t} \sim \text{i.i.d non negative variable with } E(\epsilon_{i,t}) = 1.$$

The conditional expectation of $y_{i,t}$ is denoted as: $E(y_{i,t} \mid \Omega_{i,t-1}) = \psi_{i,t} = [\psi_{i,1}, \ldots, \psi_{i,T_i}]$, where $\Omega_{i,t-1} = \{x_{i,1}, x_{i,2}, \ldots, x_{i,t-1}, y_{i,1}, y_{i,2}, \ldots, y_{i,t-1}\}$ is the past informations until the development quarter $t-1$.

This conditional expectation of claims amounts is related to the predictor $\eta_{i,t}$ by the link function such that $g(\psi_{i,t}) = \eta_{i,t}$. As pointed out by Merz and Wüthrich (2008) a log link is typically a natural choice in the insurance reserving context.

Here, the problem of temporal dependence of development quarters is considered. For that, we suppose a time dependent specification for $\eta_{i,t}$ defined by

$$\eta_{i,t} = X_{i,t}^\top \beta + \sum_{j=1}^{p} \theta_j \{g(y_{i,t-j}) - X_{i,t-j}^\top \beta\} + \sum_{j=1}^{q} \phi_j \{g(y_{i,t-j}) - \eta_{i,t-j}\}$$

where $X_{i,t}^\top$ is a $z \times p$ matrix of dummy covariates that arranges the impact of accident and development quarter on the claims amounts through model parameter $\beta \in \mathbb{R}^{p \times 1} = [\gamma, \alpha_2, \ldots, \alpha_n, \delta_2, \ldots, \delta_T]$, where $\alpha_i$ stands for the effect of accident quarter $i$, and $\delta_t$ represents the effect of the development quarter $t$, and $\gamma$ is the constant parameter of the model (taking $\alpha_1 = \delta_1 = 0$ to avoid over parametrization).
Standardized claims amounts $\epsilon_{i,t}$ are assumed to follow a Gamma distribution with $E(\epsilon_{i,t}) = 1$. Given $\Omega_{i,t-1}$, the distribution and the density function of the Gamma GACSM are presented respectively as

$$\forall i = 1, \ldots, n, \; t = 2, \ldots, T_i, \; T_i = T + 1 - i$$

$$F(y_{i,t}) = \frac{\Gamma(k, \frac{y_{i,t}}{\psi_{i,t}})}{\Gamma(k)}$$  \hspace{1cm} (4)

$$f(y_{i,t} \mid \Omega_{i,t-1}) = \frac{y_{i,t}^{k-1} k^k}{\psi_{i,t}^k \Gamma(k)} \exp\left(-\frac{y_{i,t}^k}{\psi_{i,t}}\right)$$  \hspace{1cm} (5)

where $k$ is the shape parameter, $\psi_{i,t} = \exp(\eta_{i,t})$ and $\eta_{i,t}$ is given by

$$\eta_{i,t} = X_{i,t}^T \beta + \sum_{j=1}^{p} \phi_j \{\log(y_{i,t-j}) - X_{i,t-j}^T \beta\} + \sum_{j=1}^{q} \theta_j \{\log(y_{i,t-j}) - \eta_{i,t-j}\}$$  \hspace{1cm} (6)

The log likelihood of the Gamma GACSM is then formalized as

$$L(y_{i,t} \mid \Omega_{i,t-1}) = \sum_{t=1}^{T} ((k-1) \log(y_{i,t}) + k \log(k) - k \log(\psi_{i,t}) - \log(\Gamma(k)) - \frac{y_{i,t}^k}{\psi_{i,t}})$$  \hspace{1cm} (7)

The parameters of GACSM are estimated by maximizing the log likelihood function, the procedure yields a consistent and asymptotically normal estimates.

The GACSM model allows to predict future claims amounts and future reserves for each lines of business. When claims amounts are regularly time dependent distributed, one can ask for the aggregate total reserves of lines of business. In the next section, dynamic dependence structure is proposed between two lines of business in order to evaluate the aggregate reserves and Solvency Capital Requirement (SCR).

## 3 Time varying copula model

### 3.1 Motivation for analysing the time varying copula

It is often believed that dependence between risks has an impact on the level of aggregate losses of the company. For that, the solvency 2 framework requires adequate modelling of the dependences between different lines of business. Correlation coefficient used by solvency 2 is not sufficient to describe the dependence structure, since the distribution of claims amounts in insurance is non linear and non elliptical.
Actuarial research used copulas functions as a more informative measures of dependence between claims than linear correlation. These functions proved their success in allowing a more general dependence structure, than simple linear correlation, but it seems unrealistic to treat dependence as constant. It is questionable to assume that the structure of dependence between claims is the same over years. As for instance, a car accident or climates conditions (snowy, rainy) may create a positive dependence between the two lines of business Auto damage and Auto liability. They can cause both a material damage related to the car and a human damage related to the driver. These two lines of business may be weakly dependent as well as strongly dependent conditional on the intensity of the damage caused by the accident or the intensity of the climates conditions. Also, this dependence may vary through years.

In order to capture the dynamic behaviour of dependence between these two lines of business, a time varying copulas models are considered, that allow the parameters of the copula to evolve across time, according to a dynamic process. To our knowledge is the first paper to investigate conditional copulas in insurance.

### 3.2 Conditional copulas models

To capture dependence between the two lines of business, the standardized claims amounts (residuals) obtained from GACSM estimation are considered to be modelled by dynamic copulas.

Let $\epsilon_{1, t}^i$ and $\epsilon_{2, t}^i$ denote the standardized claims amounts, classified in two run-off triangles of respectively, Auto Damage and Auto Liability, $\{\epsilon_{1, t}^i, \epsilon_{2, t}^i; i, t \in \nabla\}$, where $\nabla = \{i = 1, ..., n, t = 1, ..., T_i\}$.

$i$ referring to accident quarter and $t$ referring to development quarter with $n$ is the most recent accident quarter, and $T_i = T + 1 - i$ is the last delay quarter. Let $\Omega_{i,t-1}$ denotes the information set available at time $\{i, t - 1\}$.

In order to capture the dynamic dependence structure existing between the cells of these two triangles, we assume a conditional cumulative distribution functions of $\epsilon_{1, t}^i$ and $\epsilon_{2, t}^i$ as $F(\epsilon_{1, t}^i|\Omega_{i,t-1})$ and $G(\epsilon_{2, t}^i|\Omega_{i,t-1})$, respectively.

Therefore, the conditional copula function, denoted as $C_{i,t}(u_{i,t}, v_{i,t}|\Omega_{i,t-1})$ is defined by the two cumulative distribution functions of random variables $u_{i,t}$ and $v_{i,t}$.

$u_{i,t} = F(\epsilon_{1, t}^i|\Omega_{i,t-1})$ and $v_{i,t} = G(\epsilon_{2, t}^i|\Omega_{i,t-1})$ are the probability integral transform of respectively $\epsilon_{1, t}^i$ and $\epsilon_{2, t}^i$. They are known to have the uniform distributions, regardless of the original distributions, $F$ and $G$. 
To define the dynamic copula function, a conditional version of the Sklar theorem is proposed by Pattno (2001). It is given as

**Théorème 3.1** Let \( H \) be a conditional bivariate distribution function with continuous margins \( F \) and \( G \), and let \( \Omega_{i,t-1} \) be some conditioning set, then there exists a unique conditional copula \( C : [0, 1] \times [0, 1] \to [0, 1] \) such that

\[
H(\epsilon_{1,t}^1, \epsilon_{1,t}^2 | \Omega_{i,t-1}) = C(F(\epsilon_{1,t}^1 | \Omega_{i,t-1}, G(\epsilon_{1,t}^2 | \Omega_{i,t-1}) | \Omega_{i,t-1}) \forall \epsilon_{1,t}, \epsilon_{2,t} \in \mathbb{R}
\]  

Conversely, if \( C \) is a conditional copula and \( F \) and \( G \) are the conditional distribution functions of \( \epsilon_{1,t}^1 \) and \( \epsilon_{2,t}^2 \) respectively, then the function \( H \) is a bivariate conditional distribution function with margins \( F \) and \( G \).

The bivariate conditional density function of \( \epsilon_{1,t}^1 \) and \( \epsilon_{2,t}^2 \) can be constructed by the product of their copula density and their two marginal conditional densities, respectively denoted by \( f \) and \( g \):

\[
h(\epsilon_{1,t}^1, \epsilon_{1,t}^2 | \Omega_{i,t-1}) = c(u_{i,t}, v_{i,t} | \Omega_{i,t-1}) \times f(\epsilon_{1,t}^1 | \Omega_{i,t-1}) \times g(\epsilon_{2,t}^1 | \Omega_{i,t-1})
\]  

where

\[
c(u_{i,t}, v_{i,t} | \Omega_{i,t-1}) = \frac{\partial^2 C(u_{i,t}, v_{i,t} | \Omega_{i,t-1})}{\partial u_{i,t} \partial v_{i,t}},
\]

\( f(\epsilon_{1,t}^1 | \Omega_{i,t-1}) \) is the conditional density function of \( \epsilon_{1,t}^1 \) and \( g(\epsilon_{2,t}^2 | \Omega_{i,t-1}) \) is the conditional density function of \( \epsilon_{2,t}^2 \).

In the insurance context, Frees and Valdez (1998) provide a number of parametric static copulas. In the present work, various time varying parametric copulas are considered: Specifically, time varying Normal copula, time varying Clayton copula, time varying symmetrised Joe-Clayton Copula and time varying Gumbel copula.

**Time varying Normal copula**

This copula is derived from the bivariate normal distribution and is defined by

\[
C(u_{i,t}, v_{i,t}) = \frac{1}{2 \pi \sqrt{1 - \rho_t^2}} \exp\left\{ -\frac{y_{i,t}^2 - 2\rho_t y_{i,t} z_{i,t} + z_{i,t}^2}{2(1 - \rho_t^2)} \right\} dy_{i,t} dz_{i,t}
\]  

where \( \rho_t \) is the correlation coefficient that follows a time varying process

\[
\rho_t = \Lambda(\omega + \beta \rho_{t-1} + \alpha | u_{i,t-1} - v_{i,t-1})
\]  

7
where $\Lambda(x)$ is defined as $(1 - e^{-x})(1 + e^{-x})$: is the modified logistic transformation to keep $\rho_t$ in $(-1, 1)$ all times. This copula does not have a tail dependence, $\lambda^U_t = \lambda^L_t = 0$.

**Time varying Clayton copula**

This copula models positive dependence. It represents the risks which are more concentrated in the lower tail, so it correlates small losses. It is defined by

$$C(u_{i,t}, v_{i,t}) = (u_{i,t}^{\theta_t} + v_{i,t}^{\theta_t} - 1)^{-1/\theta_t} \quad \theta_t \in [-1, \infty) \setminus \{0\}$$  \hspace{1cm} (12)

Time-varying dependence processes for the Clayton copula is described as

$$\theta_t = (\omega + \beta \theta_{t-1} + \alpha \mid u_{i,t-1} - v_{i,t-1} \mid)$$  \hspace{1cm} (13)

This copula has lower tail dependence $\lambda^L_t = 2^{\theta_t^{-1}}$ and $\lambda^U_t = 0$.

**Time varying symmetrised Joe-Clayton copula**

The symmetrized Joe-Clayton copula introduced by [Patton (2006)](https://doi.org/10.1111/1467-9469.00283) is a flexible two-parameters copula, that is parametrized in terms of $\lambda^U$ and $\lambda^L$. A time-varying version of this copula, as considered in [Patton (2006)](https://doi.org/10.1111/1467-9469.00283), allows for changing degrees of asymmetry, as well as a time-varying overall level of dependence. It is defined as

$$C(u_{i,t}, v_{i,t}) = 0.5(C_{JC}(u_{i,t}, v_{i,t}) + (C_{JC}(1 - u_{i,t}, 1 - v_{i,t} + u_{i,t} + v_{i,t} - 1)$$  \hspace{1cm} (14)

where $C_{JC}$ is the Joe-Clayton copula, also called BB7, given by

$$C(u_{i,t}, v_{i,t}) = 1 - \left(1 - \left\{1 - (1 - u_{i,t})^{a_t} \right\}^{-b_t} \right) + \left[1 - (1 - v_{i,t})^{a_t} \right]^{-b_t} - 1$$  \hspace{1cm} (15)

with $a_t = \frac{1}{\text{Log}(2 - \lambda^U_t)}$ and $b_t = \frac{-1}{\text{Log}(\lambda^L_t)}$, $\lambda^U_t, \lambda^L_t \in (0, 1)$

The time dynamics equations for the parameters $\lambda^U_t$ and $\lambda^L_t$ are

$$\lambda^U_t = \Lambda(\omega_U + \beta_U \lambda^U_{t-1} + \alpha \mid u_{i,t-1} - v_{i,t-1} \mid)$$  \hspace{1cm} (16)

$$\lambda^L_t = \beta(\omega_L \beta_L \lambda^L_{t-1} + \alpha \mid u_{i,t-1} - v_{i,t-1} \mid)$$  \hspace{1cm} (17)

where $\Lambda(x) = (1 - e^{-x})^{-1}$. 

8
Time varying Gumbel copula

This copula models a positive dependence and represents the risks which are more concentrated in the upper tail. It is defined as

\[
C(u_{i,t}, v_{i,t}) = \exp\{\left[-\left(-\ln u_{i,t}\right)^{\theta_t} + \left(-\ln v_{i,t}\right)^{\theta_t}\right]^{1/\theta_t}\} \quad \theta_t \geq 1
\]  

(18)

The dynamic process for the dependence parameter is given

\[
\theta_t = (\omega + \beta\theta_{t-1} + \alpha | u_{i,t-1} - v_{i,t-1} |)
\]  

(19)

where \(\Lambda(x) = (1 - e^{-x})^{-1}\).

Time varying Gumbel copula has an upper tail dependence \(\lambda_t^U = 2 - 2^{1/\theta_t}\) and \(\lambda_t^L = 0\).

For these copulas functions, dynamic process allows dependence between claims of the cells of triangles. Past information is taken from claims of the same accident quarter, that means between development quarters, and we assume that claims from different accident quarters are independent.

3.3 Copula estimation

For parameter estimation, we are based on Canonical Maximum Likelihood (CML) method. This method uses empirical probability integral transform in order to obtain the uniform marginals needed to estimate the copula parameter. Copula parameter can then be estimated by maximizing the log likelihood function of the copula density using the uniform variables given by

\[
L_c = \frac{1}{T} \sum_{t=1}^{T} \log(c(\hat{F}(\epsilon_{1,t}^i), \hat{G}(\epsilon_{2,t}^i))) = \frac{1}{T} \sum_{t=1}^{T} \log(c(\hat{u}_{i,t}, \hat{v}_{i,t}))
\]  

(20)

3.4 Simulation method

In order to evaluate the Solvency Capital, the whole predictive distribution of reserves is needed. For that, a simulation procedure that allows to predict reserves taking into account the dynamic dependence between the two lines of business is presented below:

**Step 1:** For the two run-off triangles, parameters are estimated by maximizing the log likelihood function of the Gamma GACSM model.
Step 2: Using parameters estimates from step 1, future claims amounts of the lower unobserved triangle are predicted, we obtain $\hat{\psi}_1^{i,t}$ and $\hat{\psi}_2^{i,t}$, respectively for Auto Damage and Auto liability.

where:

$$\hat{\psi}_k^{i,t} = \exp(\hat{\eta}_k^{i,t}), \forall k = 1, 2$$
$$\hat{\eta}_k^{i,t} = X_{i,t}^{'} \hat{\beta} + \sum_{j=1}^{p} \hat{\phi}_j \{ \log(y_{i,t-j}) - X_{i,t-j}^{'} \hat{\beta} \} + \sum_{j=1}^{q} \hat{\theta}_j \{ \log(y_{i,t-j}) - \hat{\eta}_{i,t-j} \}.$$  

Step 3: We simulate the joint vectors $(u_{i,t}, v_{i,t})$ from the subjacent copula of the lower unobserved triangle, $\left\{ u_{i,t}, v_{i,t}; i,t \in \Delta \right\}$, where $\Delta = \{ i = 2, ..., n; t = n - i + 2, ..., T_i \}$.

Step 4: Residuals vectors $(\hat{\epsilon}_1^{i,t}, \hat{\epsilon}_2^{i,t})$ can be obtained as $\hat{\epsilon}_1^{i,t} = F^{-1}(u_{i,t})$ and $\hat{\epsilon}_2^{i,t} = G^{-1}(v_{i,t})$, where $F$ and $G$ are the estimated standardized claims amounts distributions.

The aggregate future claims reserves for each origin quarter $i$ and the aggregate total reserves are respectively

$$\hat{R}_i = \sum_{t \in \Delta_i} (\hat{y}_1^{i,t} + \hat{y}_2^{i,t}) \quad (21)$$
$$\hat{R} = \sum_{i=1}^{n} \hat{R}_i \quad (22)$$

Step 6: Repeat step 3, step 4 and step 5 $B$ times to obtain the aggregate total reserves distribution.

---

1The subjacent copula is the best copula for the data: we estimate parameters of various copulas. Based on the information criteria and a test of copula, we are able to select the best copula.
4 Empirical results

4.1 Marginal modelling

Data were supplied by a French insurance company and consist of incremental claims amounts of two lines of business, which are Auto Damage and Auto Liability, over the period of January 2002, to December 2007.

The motivation for considering these two lines of business is that they present a large number of claims in the insurance company, and they are highly dependent. The first guarantee covers all damages caused to the car, even if the insured is responsible for the accident. It is a short settlement line of business, that once the claim is declared it will be paid after a short time.

The second is the guarantee of damage caused to others, belongs to the Civil Responsibility lines of business. It is called a long settlement line of business, because there will be a time lag between the date of occurrence claim and the date of a full refund. These claims amounts are classified quarterly into two run-off triangles. We have 300 observations for each lines of business spread over 24 accident quarters and 24 development quarters. The claims amounts are net of reinsurance, net of subrogation and we suppose that inflation remains unchanged throughout the quarters. To investigate the characteristics of these claims, summary statistics of claims amounts of the two lines of business are presented in Table 1.

<table>
<thead>
<tr>
<th></th>
<th>Auto damage</th>
<th>Auto liability</th>
</tr>
</thead>
<tbody>
<tr>
<td>max</td>
<td>517.129</td>
<td>497.472</td>
</tr>
<tr>
<td>min</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>mean</td>
<td>64.131</td>
<td>62.670</td>
</tr>
<tr>
<td>std</td>
<td>115.165</td>
<td>96.369</td>
</tr>
<tr>
<td>skewness</td>
<td>2.005</td>
<td>2.097</td>
</tr>
<tr>
<td>kurtosis</td>
<td>6.028</td>
<td>6.807</td>
</tr>
<tr>
<td>Jarque Bera test</td>
<td>315.7 (0.001)</td>
<td>401.02 (0.001)</td>
</tr>
</tbody>
</table>

( . ) contains p-value of the Jarque Bera test

For the lines Auto Damage and Auto Liability, the minimum and maximum claims amounts suggest that the distributions are widely spread. The standard deviation is larger than the mean, implying the skewness of the data. In fact, the two distributions of claims amounts exhibit a positive skewness and excess kurtosis indicating that are heavy tailed and right skewed.
According to Jarque Bera test, there is evidence that the two distributions of claims amounts are not normally distributed.

The ACF plot (Figure 5) presented in the Appendix A shows a significant lags, indicating that claims amounts exhibit temporal correlations. These correlations are positive, indicating that large or small claims amounts tend to occur together.

For these two lines of business, the GACSM model is fitted. Results are reported in Table 6 and Table 7 in Appendix B.

From Table 6 and Table 7 in Appendix B, we remark that parameters estimates are significantly different from zero. Moreover, from Table 5 there is no autocorrelation in the residuals as shown by the statistics of the Ljung Box test. Also, Figure 6 and Figure 7 in the Appendix C show that the QQplot of residuals of the Auto Damage and Auto Liability versus respectively the Gamma (0.367,2.75) and Gamma (0.527,1.9) distributions are very close. Thus the two lines of business are well specified.

These two lines of business present a Pearson’s correlation coefficient equal to 0.93, they are highly dependent. Therefore, a car accident can affect the line Auto Damage as well as the line Auto Liability. But, one can ask if this linear correlation is constant over time. For that, in the next section the two marginals of residuals obtained from the estimation with GACSM are estimated with a dynamic copula model, which can capture a non linear as well as a time varying dependence structure.

4.2 Modelling copulas

For the lines of business Auto Damage and Auto Liability, the standardized claims amounts (residuals) are transformed into uniform marginals using the empirical probability integral transform. The log likelihood function of each copula density is maximized to obtain parameters estimates. (All estimates of copulas functions were made using the copula toolbox provided by Andrew Patton (University of Oxford-Department of economics)) and some functions created by authors on MATLAB 7. Results are reported in Table 2. Figure 1 illustrates the time varying parameters estimates corresponding to each copula function.

As we can see from Table 2 parameters estimates of the time varying copulas are statistically different from zero. This indicates that the dependence between the two lines of business changes in time. This dynamic behavior of dependence is captured by the specific properties of each copula function (equations (11), (13), (16), (17) and (19)).
Figure 1: Time varying parameters estimates corresponding to Normal, Clayton, SJC and Gumbel copulas
Table 2: Parameters estimates of time varying copulas (tvar). Log likelihood function (LL), AIC and BIC criteria.

<table>
<thead>
<tr>
<th></th>
<th>tvar Normal</th>
<th>tvar Clayton</th>
<th>tvar SJC</th>
<th>tvar Gumbel</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega )</td>
<td>0.035 [31.36]***</td>
<td>0.382 [62.4]***</td>
<td>0.932 [3.96]**</td>
<td>-2.820 [12.96.6]***</td>
</tr>
<tr>
<td>( \beta )</td>
<td>0.093 [10.24]**</td>
<td>-1.723 [5.76]**</td>
<td>-24.915 [3.92]**</td>
<td>-0.870 [73.96]***</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>1.809 [29.7]***</td>
<td>0.698 [5.29]**</td>
<td>-5.565 [10.3]***</td>
<td>8.218 [15.3]***</td>
</tr>
<tr>
<td>LL</td>
<td>7.581</td>
<td>11.726</td>
<td>13.469</td>
<td>15.498</td>
</tr>
</tbody>
</table>

( . ) contains the corresponding Wald test statistics,

**: 5% significant statistics,

**: 1% significant statistics.

From Figure 1, Normal copula panel (a) displays a dynamic correlation coefficient that shows a significant dynamic dependence between claims amounts. The value of the constant parameter is 0.194 and time evolution takes place around this value and becomes almost constant over the recent period.

Clayton copula in Panel (b), which measures the dependence in the lower tail, captures a constant coefficient of 0.289. The time evolution of the parameter fluctuates almost below the constant and equals even zero during certain periods. This result shows that small amounts of the two lines of business are weakly dependent. A small car damage does not lead necessarily to a claims in the line Auto Liability.

This behavior is also captured by the SJC copula, which measures dependence both in the lower and upper tail. The coefficient of lower tail dependence is 0.0156. The evolution in time of this parameter remains around this value during the period of study, except for recent period when it raises and achieves 0.972. This result is informative, showing certain standard behavior for the lower dynamic dependence of small claims amounts. The coefficient of upper tail dependence for SJC copula is shown in Panel (d). Constant coefficient of upper tail is equal to 0.1403 higher than the lower tail. This result is confirmed by the evolution in time of this parameter. The upper tail exhibits a high time varying dependence between claims amounts that achieves to 0.7.

Gumbel copula captures also a highly time varying dependence that evolves from 1 to 2.28, indicating that large claims amounts are strongly dependent. This means that a car accident can induce a large claims amounts into the line Auto Damage as well as into the line Auto Liability, as it causes a material damage, and it can frequently cause physical injuries related to the civil responsibility.
To choose the best copula model to our data, we are based on the loglikelihood function ($LL$), AIC and BIC criteria. From Table 2 we can conclude that the Gumbel time varying copula model presents a higher $LL$ and a lower AIC and BIC values. Hence, this model gives a better fit for data than the other models and this copula provides a more accurate description of claims amounts behavior.

To validate this choice, a Vuong test is performed for each pair of copula. This test is based on the likelihood ratio and is proposed by [Vuong (1989)] for non nested models (see [Vuong (1989)] for more details on the Vuong test). This test is appropriate as our copula models are non nested. Table 3 displays the results of the Vuong test.

<table>
<thead>
<tr>
<th>Model A</th>
<th>tvar Gumbel</th>
<th>tvar SJC</th>
<th>tvar Clayton</th>
<th>tvar Normale</th>
</tr>
</thead>
<tbody>
<tr>
<td>tvar Gumbel</td>
<td>(1.41)</td>
<td>tvar Gumbel (4.28)</td>
<td>tvar Gumbel (5.9)</td>
<td></td>
</tr>
<tr>
<td>tvar SJC</td>
<td>(4.93)</td>
<td>tvar SJC (4.47)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>tvar Clayton</td>
<td></td>
<td>tvar Clayton (-1.4)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>tvar Normale</td>
<td></td>
<td></td>
<td></td>
<td>tvar Normale</td>
</tr>
</tbody>
</table>

Table 3: Pairwise Vuong tests. We display the copula family that is selected on a $\alpha = 0.05$ level. In parentheses, we display the value of the Vuong test statistic.

For each pair, we present the copula family that is selected on a $\alpha = 0.05$ level. In parentheses, we display the value of the Vuong test statistics. Note that a statistic value $> 2$ indicates that we select model A, and a value $< -2$ indicates that we select model B. Otherwise, no decision among the two copula families is possible. As shown in Table 3, the tvar Gumbel copula model significantly outperforms the tvar Clayton and the tvar Normal copulas models. But, no decision among the tvar Gumbel and the SJC copulas could be made. So, there is no substantial difference between these two copulas models.

Based on results reported in Table 2 ($LL$, AIC, BIC), tvar Gumbel copula is preferred to SJC copula model. Therefore, in the following section, we continue our analysis with the tvar Gumbel copula model.

\[ H_0: \frac{LR(\hat{\beta}, \hat{\gamma})}{n^{1/2} \hat{\omega}} \]

\[ LR(\hat{\beta}, \hat{\gamma}) = \ln f(y|x; \hat{\beta}) - \ln g(y|x; \hat{\gamma}) \]

\[ \hat{\omega}^2 = \sum_{i=1}^{n} \left[ \frac{\ln f(y|x; \hat{\beta})}{g(y|x; \hat{\gamma})} \right]^2 - \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{\ln f(y|x; \hat{\beta})}{g(y|x; \hat{\gamma})} \right]^2 \]
4.3 Impact of time varying dependence on total loss

To investigate the impact of time varying dependence on the evaluation of reserves and solvency capital, a simulation study is conducted. 1000 replications of the lower unobserved triangle are drawn from the tvar Gumbel copula to the two lines of business as in [3,4]. The simulated claims are summed up to an aggregated total reserves distribution as in equation 22.

Then, some quantities are computed on these aggregate total reserves: Mean, Value at Risk ($VaR_{99.5\%}$), Conditional Value at Risk ($CVaR_{99.5\%}$) and Solvency Capital Requirement ($SCR_3$).

For comparison purposes, we present below the results of modelling with the static Symmetrized Joe Clayton copula model [1], Generalized Autoregressive Conditional sinistrality model and the classical Chain Ladder model (CL). Indeed, data are fitted, and aggregate total reserves distributions are predicted under each model.

We specify that model (1) is the time varying Gumbel copula model with GACSM marginals.

Model (2) considers that the two marginals are modelled and predicted with a GACSM model, and the joint distribution of reserves is predicted from a SJC static copula, where the dependence structure is constant over time. (simulation procedure for this copula is the same as in [3,4].

Model (3) is the GACSM without copula. The two marginals are predicted independently using the GACSM model and the joint distribution is the sum of the two marginals distributions. Indeed, temporal dependence structure is identified among claims amounts of each line of business, and we assume that claims amounts between the two lines of business are independent.

---

$^{3}$Solvency Capital Requirement of reserve risk for a line of business $l$, is given by

$$SCR_l = VaR_{99.5\%}^l - BE_l$$

(23)

where $VaR_{99.5\%}^l$ is the Value at Risk of the line business $l$ at confidence level 99.5%, and $BE_l$ is the best estimate of the total reserves of the line of business $l$, computed as the mean of the total reserves.

$^{4}$We note that we estimated four static copulas corresponding to the Normal, Clayton, Symmetrized Joe Clayton and the Gumbel copulas (more details about static copulas are presented in [D]). According to LL, AIC and BIC criteria the best copula was the Symmetrized Joe Clayton (see table 8 in Appendix D).
Model (4) is the Chain Ladder model (CL) usually considered in the literature as the benchmark model. Here, a classical bootstrap version of Chain Ladder is conducted to obtain the total reserves distributions (see England and Verrall (2002)).

The Results of the proposed approach time varying copula model and the copula static model, the Generalized Conditional Sinistrality model and the traditional Chain Ladder model are compared numerically in Table 4 and graphically in Figures 2, 3 and 4.

<table>
<thead>
<tr>
<th>tvar Gumbel</th>
<th>static SJC</th>
<th>GACSM</th>
<th>CL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>2569.867</td>
<td>2449.716</td>
<td>2780.678</td>
</tr>
<tr>
<td>VaR99.5%</td>
<td>3903.742</td>
<td>3575.782</td>
<td>4478.167</td>
</tr>
<tr>
<td>CVaR99.5%</td>
<td>4185.886</td>
<td>3746.795</td>
<td>4603.496</td>
</tr>
<tr>
<td>SCR</td>
<td>1333.876</td>
<td>1126.066</td>
<td>1697.489</td>
</tr>
</tbody>
</table>

Figure 2: Scatter plots of the aggregate total reserves
Figure 3: Total aggregate reserves distributions from different models

Figure 4: Total aggregate reserves distributions
It is clearly visible from Figure 2 that the three scatter plots of models (1), (2) and (3), considerably differ in range and shape from model (4). Total reserves under these models are more stretched than total reserves under model (4), when each line of business has been modelled independently of the other line.

Indeed, in the tvar Gumbel case the scatter plot shows a dependence in the upper right corner, while in the SJC case there is a reasonable dependence both in the lower and upper corner. In contrast to these models, Chain Ladder model presents a random pattern of the pairs of total reserves between the two business lines.

Further features are captured by the cumulative and density distributions of Figure 3. As we can see, predictive distributions of the aggregate reserves from Chain Ladder model is more tighter than other models. This is explained by the fact that this method ignores the dependence structure both among claims amounts and between the two lines. When modelling with GACSM, this model provides a fatter aggregate reserves distribution than copula models.

Thus, considering the association between the two lines leads to estimate an aggregate total reserves lower than when we treat the two lines of business as unrelated.

In addition, Figure 4 displays a right long tail distributions of the aggregate reserves under tvar Gumbel and SJC copula models. This is explained by the fact that these two copulas models take into account the dependence structure into the upper tail of the reserves distribution.

To show numerically the effect of dependence structure, some quantities are computed on the aggregate total reserves distributions for each model. Results are summarized in Table 4.

At first sight, the results shed some light on the practical effects of dependence modelling. Specifically, tvar Gumbel copula provides a total aggregate amount of reserves significantly lower than the GACSM model and the Chain Ladder model. However, it provides a 4.9% higher amount of reserves than the SJC model. Hence, a diversification effect could be gained on the amount of reserves when considering copula models. But, when we analyse the effect of the tvar Gumbel on the measures of risk, this copula produces considerably a higher $VaR_{99.5\%}$ and $CVaR_{99.5\%}$ and therefore a significantly higher SCR than the SJC copula and the Chain Ladder model. This explained by the fact that this copula takes into account the time dependence structure of risks, which are more concentrated in the upper tail, and therefore, it generates in additional amount of capital.

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Moreover, SCR under tvar Gumbel copula is 21.42% lower than SCR under GACSM. Hence a diversification effect could be gained when we consider a time varying dependence structure between the two lines of business. However, SCR under this copula is 18.5% and 20.47% higher than SCR evaluated respectively under SJC copula model and Chain Ladder model. Hence, time varying dependence structure generate in additional amount of capital, contrary to the constant dependence structure that can induce an underestimation of risk. This effect is even more pronounced for the Chain Ladder model which ignores dependence among claims amounts as well between lines of business. So, this model could underestimate the risk.

5 Conclusion

Prior actuarial works have suggested a constant dependence structure to accommodate relation between lines of business. In this paper a time varying dependence structure was proposed to model dependencies between claims amounts of two lines of business. Such framework was demonstrated to be suitable for stochastic claims reserving of non life insurance.

The proposed model takes the advantage on traditional methods for being dynamic and it captures the evolution in time of the dependence structure. Indeed, contrary to classical reserving techniques, a temporal dependence structures are captured both between claims amounts of each line of business and between claims of the two lines of business. In this regard, GACSM enables us to specify a temporal dependence between development quarters of a line of business, and the time varying Gumbel copula model allows to capture the dynamic behavior of dependence between the two cells of triangles.

For these two lines of business a simulation study was performed to show the implication of such dependencies on predictive reserves and some measures such that VaR, CVaR and SCR. The time varying Gumbel copula with the GACSM provide a significant lower total reserves and SCR than the GACSM without copula. Therefore, we can conclude that this dynamic behaviour of the structure of dependence provides a diversification effect between the two lines, and a potential benefit for in insurance company may be gained by reducing the cost of capital. In this regard, insurer might consider expanding the Auto Damage or shrinking the Auto Liability to take best advantage of the diversification effect.
Appendix

Autocorrelations of the line Auto Damage and Auto liability

Figure 5: Autocorrelations of the line Auto Damage and Auto liability

<table>
<thead>
<tr>
<th></th>
<th>Auto Damage</th>
<th>Auto Liability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.996</td>
<td>0.999</td>
</tr>
<tr>
<td>Variance</td>
<td>1.454</td>
<td>1.1</td>
</tr>
<tr>
<td>LJB test Q(10)</td>
<td>11.02 (0.356)</td>
<td>15.05 (0.13)</td>
</tr>
<tr>
<td>LM test Q(10)</td>
<td>2.079 (0.996)</td>
<td>1.15 (0.999)</td>
</tr>
</tbody>
</table>

Results of estimating marginals
Table 6: Average values and Standard errors of the estimated parameters of computed from 1000 iterations for GACSM Gamma and GLM Gamma

<table>
<thead>
<tr>
<th>Parameter</th>
<th>GACSM Gamma</th>
<th>GLM Gamma</th>
<th>GACSM Gamma</th>
<th>GLM Gamma</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>mean</td>
<td>std</td>
<td>mean</td>
<td>std</td>
</tr>
<tr>
<td>$k$</td>
<td>0.367***</td>
<td>0.027</td>
<td>2.697***</td>
<td>0.197</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.039*</td>
<td>0.021</td>
<td>0.086*</td>
<td>0.017</td>
</tr>
<tr>
<td>$\theta$</td>
<td>-0.041*</td>
<td>0.026</td>
<td>-0.092*</td>
<td>0.049</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>12.309***</td>
<td>0.383</td>
<td>12.459***</td>
<td>0.298</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>-0.066*</td>
<td>0.036</td>
<td>-0.072*</td>
<td>0.040</td>
</tr>
<tr>
<td>$\alpha_3$</td>
<td>-0.347*</td>
<td>0.210</td>
<td>-0.438*</td>
<td>0.265</td>
</tr>
<tr>
<td>$\alpha_4$</td>
<td>-0.210*</td>
<td>0.125</td>
<td>-0.073*</td>
<td>0.045</td>
</tr>
<tr>
<td>$\alpha_5$</td>
<td>0.097*</td>
<td>0.054</td>
<td>0.057*</td>
<td>0.037</td>
</tr>
<tr>
<td>$\alpha_6$</td>
<td>-0.066*</td>
<td>0.037</td>
<td>-0.085*</td>
<td>0.044</td>
</tr>
<tr>
<td>$\alpha_7$</td>
<td>-0.185*</td>
<td>0.099</td>
<td>-0.057*</td>
<td>0.034</td>
</tr>
<tr>
<td>$\alpha_8$</td>
<td>-0.030*</td>
<td>0.017</td>
<td>-0.020*</td>
<td>0.018</td>
</tr>
<tr>
<td>$\alpha_9$</td>
<td>-0.032*</td>
<td>0.017</td>
<td>-0.035*</td>
<td>0.019</td>
</tr>
<tr>
<td>$\alpha_{10}$</td>
<td>-0.159*</td>
<td>0.087</td>
<td>-0.025*</td>
<td>0.015</td>
</tr>
<tr>
<td>$\alpha_{11}$</td>
<td>-0.087*</td>
<td>0.05</td>
<td>-0.049*</td>
<td>0.025</td>
</tr>
<tr>
<td>$\alpha_{12}$</td>
<td>-0.028*</td>
<td>0.016</td>
<td>-0.026*</td>
<td>0.015</td>
</tr>
<tr>
<td>$\alpha_{13}$</td>
<td>0.229</td>
<td>0.135</td>
<td>0.101*</td>
<td>0.061</td>
</tr>
<tr>
<td>$\alpha_{14}$</td>
<td>-0.070*</td>
<td>0.041</td>
<td>-0.072*</td>
<td>0.038</td>
</tr>
<tr>
<td>$\alpha_{15}$</td>
<td>-0.151*</td>
<td>0.082</td>
<td>-0.116*</td>
<td>0.064</td>
</tr>
<tr>
<td>$\alpha_{16}$</td>
<td>0.113</td>
<td>0.062</td>
<td>0.128*</td>
<td>0.078</td>
</tr>
<tr>
<td>$\alpha_{17}$</td>
<td>0.217*</td>
<td>0.125</td>
<td>0.132*</td>
<td>0.071</td>
</tr>
<tr>
<td>$\alpha_{18}$</td>
<td>-0.055*</td>
<td>0.031</td>
<td>-0.058*</td>
<td>0.035</td>
</tr>
<tr>
<td>$\alpha_{19}$</td>
<td>-0.053*</td>
<td>0.030</td>
<td>-0.055*</td>
<td>0.029</td>
</tr>
<tr>
<td>$\alpha_{20}$</td>
<td>0.069*</td>
<td>0.04</td>
<td>0.033*</td>
<td>0.014</td>
</tr>
<tr>
<td>$\alpha_{21}$</td>
<td>0.208*</td>
<td>0.124</td>
<td>0.121*</td>
<td>0.071</td>
</tr>
<tr>
<td>$\alpha_{22}$</td>
<td>-0.138*</td>
<td>0.081</td>
<td>-0.124*</td>
<td>0.072</td>
</tr>
<tr>
<td>$\alpha_{23}$</td>
<td>-0.293*</td>
<td>0.165</td>
<td>-0.188*</td>
<td>0.116</td>
</tr>
<tr>
<td>$\alpha_{24}$</td>
<td>-4.895*</td>
<td>2.951</td>
<td>-4.914*</td>
<td>2.815</td>
</tr>
</tbody>
</table>

Wald test is applied to test whether estimated parameter is 0.

*: 10% significant statistics.
**: 5% significant statistics.
***: 1% significant statistics.
Table 7: Average values and standard errors of the estimated development quarter parameters, computed from 1000 iterations for GACSM Gamma and GLM Gamma

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Damage Liability</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>GACSM Gamma</td>
</tr>
<tr>
<td>δ2</td>
<td>0.434* 0.217</td>
</tr>
<tr>
<td>δ3</td>
<td>-0.772* 0.463</td>
</tr>
<tr>
<td>δ4</td>
<td>-1.776*** 0.473</td>
</tr>
<tr>
<td>δ5</td>
<td>-2.122*** 0.483</td>
</tr>
<tr>
<td>δ6</td>
<td>-2.859*** 0.489</td>
</tr>
<tr>
<td>δ7</td>
<td>-2.852*** 0.497</td>
</tr>
<tr>
<td>δ8</td>
<td>-3.444*** 0.497</td>
</tr>
<tr>
<td>δ9</td>
<td>-2.985*** 0.518</td>
</tr>
<tr>
<td>δ10</td>
<td>-4.827*** 0.519</td>
</tr>
<tr>
<td>δ11</td>
<td>-5.717*** 0.521</td>
</tr>
<tr>
<td>δ12</td>
<td>-5.305*** 0.551</td>
</tr>
<tr>
<td>δ13</td>
<td>-2.827*** 0.546</td>
</tr>
<tr>
<td>δ14</td>
<td>-5.768*** 0.554</td>
</tr>
<tr>
<td>δ15</td>
<td>-6.130*** 0.588</td>
</tr>
<tr>
<td>δ16</td>
<td>-5.088*** 0.593</td>
</tr>
<tr>
<td>δ17</td>
<td>-5.517*** 0.625</td>
</tr>
<tr>
<td>δ18</td>
<td>-12.232*** 0.672</td>
</tr>
<tr>
<td>δ19</td>
<td>-12.459*** 0.730</td>
</tr>
<tr>
<td>δ20</td>
<td>-12.516*** 0.763</td>
</tr>
<tr>
<td>δ21</td>
<td>-8.013*** 0.865</td>
</tr>
<tr>
<td>δ22</td>
<td>-4.610*** 1.161</td>
</tr>
</tbody>
</table>

Wald test is applied to test whether estimated parameter is 0.

*: 10% significant statistics.

**: 5% significant statistics.

***: 1% significant statistics.
C QQ plots

Figure 6: QQplot Auto damage residuals versus Gamma (0.367,2.75)

Figure 7: QQplot Auto liability residuals versus Gamma (0.527,1.9)
D Copulas

In this section, we present statistic copulas for dependent claims amounts relative to the two lines of business. Copulas functions have been introduced in the insurance context by Frees and Valdez (1998). A copula is based on an assumption that both marginal distributions are known. For more details about copula functions, we refer readers to Nelsen (2006).

Let \( F_X(x) \) and \( F_Y(y) \) denote the marginal distribution functions of the variables \( X \) corresponding to the claims amounts of the line Auto Damage and \( Y \) corresponding to the claims amounts of the line Auto Liability.

The joint distribution function \( F_{X,Y}(x, y) \) is then obtained as

\[
F_{X,Y}(x, y) = C[F_X(x), F_Y(y)]
\]  

(24)

where \( C(u, v) \) is the copula, a cumulative distribution function for a bivariate distribution with support on the unit square and uniform marginals.

In this paper, we assume that the marginal distributions are continuous with density functions \( f_X(x) \) and \( f_Y(y) \). Then, the joint density function is

\[
f_{X,Y}(x, y) = f_X(x)f_Y(y)C_{12}[f_X(x), f_Y(y)]
\]  

(25)

where

\[
C_{12}(u, v) = \frac{\partial^2 C(u, v)}{\partial u \partial v}
\]

The conditional distribution function of \( Y \mid X = x \) is

\[
F_{Y \mid X}(y \mid x) = C_1[F_X(x), F_Y(y)]
\]  

(26)

where

\[
C_1(u, v) = \frac{\partial C(u, v)}{\partial u}
\]

In the insurance context, Frees and Valdez (1998) provide a number of copulas. In each case, the parameter \( \alpha \) is a constant parameter that measures the degree of association. In the present work, we used several static copulas in order to select the appropriate one, that describe the structure of dependence between the two lines of business.

We investigate Normal copula, Clayton copula, Symmetrized Joe Clayton copula and Gumbel copula.
Normal copula

This copula is derived from the bivariate normal distribution and is defined by

\[ C(u_{i,t}, v_{i,t}) = \int_{-\infty}^{\Phi^{-1}(u_{i,t})} \int_{-\infty}^{\Phi^{-1}(v_{i,t})} \frac{1}{2\pi \sqrt{1 - \rho^2}} \exp\left\{ -\frac{y_{i,t}^2 - 2\rho y_{i,t}z_{i,t} + z_{i,t}^2}{2(1 - \rho^2)} \right\} \, dy_{i,t} \, dz_{i,t} \tag{27} \]

where the correlation coefficient \( \rho \in (-1, 1) \).

Clayton

This copula models positive dependence, it represents the risks which are more concentrated in the lower tail, so it correlates small losses. It is defined as

\[ C(u, v) = (u^{-\alpha} + v^{-\alpha} - 1)^{-1/\alpha} \quad \alpha \in [-1, \infty) \setminus \{0\} \tag{28} \]

The Clayton copula has lower tail dependence \( \lambda_L = 2^{-1/\alpha} \) and \( \lambda_U = 0 \).

Symmetrised Joe-Clayton copula

The symmetrized Joe-Clayton copula introduced by \textsuperscript{[Patton (2006)]} is a flexible two-parameter copula that is parametrized in terms of \( \lambda^U \) and \( \lambda^L \). A time-varying version of this copula, as considered in \textsuperscript{[Patton (2006)]}, allows for changing degrees of asymmetry, as well as a time-varying overall level of dependence. It is defined as

\[ C(u_{i,t}, v_{i,t}) = 0.5(C_{JC}(u_{i,t}, v_{i,t}) + (C_{JC}(1 - u_{i,t}, 1 - v_{i,t}) + u_{i,t} + v_{i,t} - 1) \tag{29} \]

where \( C_{JC} \) is the Joe-Clayton copula, also called BB7, given by

\[ C(u_{i,t}, v_{i,t}) = 1 - \left( 1 - \left\{ [1 - (1 - u_{i,t})^a]^{-b} \right\} + [1 - (1 - v_{i,t})^a]^{-b} - 1 \right)^{-1/a} \tag{30} \]

where \( a = \frac{1}{\log(2 - \lambda^U)} \) and \( b = \frac{-1}{\log(\lambda^L)} \), \( \lambda^U, \lambda^L \in (0, 1) \).

Gumbel

This copula models a positive dependence and represents the risks which are more concentrated in the upper tail. It is defined as

\[ C(u, v) = \exp\{-[(lnu)^\alpha + (-lnv)^\alpha]^{1/\alpha}\} \quad \alpha \geq 1 \tag{31} \]
where $\alpha$ is the parameter dependence.

The Gumbel copula has upper tail dependence $\lambda_U = 2 - 2^{1/\alpha}$, but no lower tail dependence.

By using these copulas, we are able to detect dependence in the tails. In order to estimate copulas model, we are based on the Canonical maximum likelihood method. This method consists in transforming the data of claims amount $(x_1^t, ..., x_N^t)$ into uniform variates $(\hat{u}_1^t, ..., \hat{u}_N^t)$ using the empirical distribution functions, and then estimate the parameter in the following way

$$\hat{\alpha} = \arg\max_{\alpha} \sum_{t=1}^{T} \ln c(\hat{u}_1^t, ..., \hat{u}_N^t; \alpha)$$

(32)

The parameters of copulas are estimated by maximizing the log likelihood function of each copula, the procedure yields a consistent and asymptotically normal estimates.

Table 8: Parameters estimates of static copulas. Log likelihood function ($LL$), AIC and BIC criteria.

<table>
<thead>
<tr>
<th></th>
<th>Normale</th>
<th>Clayton</th>
<th>SJC</th>
<th>Gumbel</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa$</td>
<td>0.194 (2.04)**</td>
<td>0.289 (2.27)**</td>
<td>0.016 (3.12)**</td>
<td>0.140 (5.7)**</td>
</tr>
<tr>
<td>$LL$</td>
<td>5.777</td>
<td>6.719</td>
<td>6.955</td>
<td>1.16 (40.7)**</td>
</tr>
</tbody>
</table>

( ) contains the corresponding Student test statistics,

**: 5% significant statistics,

***: 1% significant statistics.
References


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